Remarks on Prime Ideal and Representation Theorems for Double Boolean Algebras

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Abstract. The notion of a double Boolean algebra was proposed by Wille in 2000. In 2006, Kwuida redefined this algebraic structure by adding two new axioms, retaining the same name for it. Notions of primary ideals and filters were introduced for this enhanced structure and the prime ideal theorem was proved. In this work, we show that the two axioms considered by Kwuida are derivable in Wille's double Boolean algebra. As a consequence, the prime ideal theorem holds for Wille's double Boolean algebra itself. We also discuss representation theorems for the class of double Boolean algebras, including, in particular, the result for representation of the class of regular double Boolean algebras given recently by Breckner and Săcărea.

1 Introduction

Formal concept analysis [4] was introduced by Wille and has since been successfully applied to many areas [7, 8]. In [13], the negation of a formal concept was introduced by Wille to enhance the possibility of expression of conceptual knowledge [6, 11–13]. Boole's correspondence between negation and set-complement was taken as a basis to formulate the negation. However, there turns out to be a problem of closure if set-complement is used, and the notion of concept is generalized to that of a semiconcept [13] and further to a protoconcept [14]. Semiconcepts and protoconcepts yield algebraic structures, and the protoconcept algebra leads to the notion of a double Boolean algebra (dBa) [14]. Pure dBas [14] constitute a special subclass of dBas, and are intended to reflect the semiconcept algebra. In particular, every Boolean algebra is a (pure) dBa.

Algebraic studies of dBas led to the natural question whether the important prime ideal theorem of Boolean algebras also holds for dBas. The question was addressed by Kwuida. In [5], Kwuida redefines Wille's dBa by adding two new axioms, retaining the name 'double Boolean algebra' for the new class of algebras. Notions of primary ideals and filters are introduced in [5] and a result analogous to the prime ideal theorem is proved for these enhanced structures. In this work, we show that the two axioms considered by Kwuida are derivable from the

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definition of Wille's dBas (Section 3), and observe the consequence that the prime ideal theorem proved in [5] holds for the class of Wille's dBas itself. In Section 4, we discuss existing representation theorems for dBas. In particular, we comment on the result for representation of the class of regular dBas given recently by Breckner and Săcărea in Section 4.1. We give a counterexample and note that the result holds in the special case of Boolean algebras.

Preliminaries required for the work are given in the next section. Section 5 concludes the paper.

2 Preliminaries

Definition 1. [4] A *context* is a triple $\mathbb{K} := (G, M, R)$, where G is the set of *objects*, and M is the set of *attributes*. $R \subseteq G \times M$.
For any $A \subseteq G, B \subseteq M$ the following sets are defined:
$A' := \{m \in M : \forall g \in G(g \in A \Longrightarrow gRm)\}$,
$B' := \{g \in G : \forall m \in M(m \in M \Longrightarrow gRm)\}$.
(A, B) is a *concept* of $\mathbb{K}$ provided $A' = B$ and $B' = A$.
There is a partial order relation $\leq$ on the set of all concepts defined as follows: for concepts $(A_1, B_1), (A_2, B_2)$, $(A_1, B_1) \leq (A_2, B_2)$ if and only if $A_1 \subseteq A_2$ (equivalently $B_2 \subseteq B_1$).

Notation 1 The set of all concepts of a context $\mathbb{K}$ is denoted by $\mathcal{B}(\mathbb{K})$. For a concept (A, B) of $\mathbb{K}$, $A := ext((A, B))$ is its *extent* and $B := int((A, B))$ is its *intent*.

The partial ordered set $(\mathcal{B}(\mathbb{K}), \leq)$ forms a complete lattice [3]. Moreover, every complete lattice is isomorphic to the concept lattice of some context. For further details about formal concept analysis, we refer to [3, 4].

Definition 2. Let $\mathbb{K} := (G, M, R)$ be a context and $A \subseteq G, B \subseteq M$. The pair (A, B) is called a *semiconcept* of $\mathbb{K}$ if and only if $A' = B$ or $B' = A$.
(A, B) is called a *protoconcept* of $\mathbb{K}$ if and only if $A'' = B'$.

Notation 2 The set of all semiconcepts of $\mathbb{K}$ is denoted by $\mathfrak{H}(\mathbb{K})$, while that of all protoconcepts is denoted by $\mathfrak{P}(\mathbb{K})$. Clearly, $\mathfrak{H}(\mathbb{K}) \subseteq \mathfrak{P}(\mathbb{K})$.

There is a partial order relation $\leq$ (we use the same notation as before) on the set $\mathfrak{P}(\mathbb{K})$ of protoconcepts also, defined as follows: $(A, B) \leq (C, D)$ if and only if $A \subseteq C$ and $D \subseteq B$, for all $(A, B), (C, D) \in \mathfrak{P}(\mathbb{K})$. The following operations are defined on $\mathfrak{P}(\mathbb{K})$. For (A_1, B_1) and (A_2, B_2) in $\mathfrak{P}(\mathbb{K})$,

$$\begin{aligned}
(A_1, B_1) \sqcap (A_2, B_2) &:= (A_1 \cap A_2, (A_1 \cap A_2)^{'}) \\
(A_1, B_1) \sqcup (A_2, B_2) &:= ((B_1 \cap B_2)^{'}, B_1 \cap B_2) \\
\neg(A, B) &:= (G \setminus A, (G \setminus A)^{'}) \\
\lrcorner(A, B) &:= ((M \setminus B)^{'}, M \setminus B) \\
\top &:= (G, \phi) \\
\bot &:= (\phi, M).
\end{aligned}$$

$\mathfrak{P}(\mathbb{K})$ forms an abstract algebra of type $(2, 2, 1, 1, 0, 0)$ with respect to the above operations. This algebra is called the *protoconcept algebra* of the context $\mathbb{K}$, and is denoted by $\underline{\mathfrak{P}}(\mathbb{K}) := (\mathfrak{P}(\mathbb{K}), \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$. The set $\mathfrak{H}(\mathbb{K})$ of semiconcepts is closed under the above operations and so it forms a subalgebra of the protoconcept algebra, called the *semiconcept algebra* of $\mathbb{K}$. It is denoted by $\underline{\mathfrak{H}}(\mathbb{K})$.

Observation 1 The partial order $\leq$ on $\mathfrak{P}(\mathbb{K})$ is characterized by the operations $\sqcap, \sqcup$: for all $x, y \in \mathfrak{P}(\mathbb{K})$,

$$x \leq y \text{ if and only if } x \sqcap y = x \sqcap x \text{ and } x \sqcup y = y \sqcup y.$$

On abstraction of properties of the protoconcept algebra $\underline{\mathfrak{P}}(\mathbb{K})$, the double Boolean algebra (dBa) [14] is defined, while the semiconcept algebra $\underline{\mathfrak{H}}(\mathbb{K})$ leads to the notion of a pure double Boolean algebra [14].

Definition 3. [14] The structure $\mathbf{D} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ is called a *double Boolean algebra* (dBa) if the following properties are satisfied. For any $x, y, z \in D$,

(1a) $(x \sqcap x) \sqcap y = x \sqcap y$	(1b) $(x \sqcup x) \sqcup y = x \sqcup y$
(2a) $x \sqcap y = y \sqcap x$	(2b) $x \sqcup y = y \sqcup x$
(3a) $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$	(3b) $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$
(4a) $\neg(x \sqcap x) = \neg x$	(4b) $\lrcorner(x \sqcup x) = \lrcorner x$
(5a) $x \sqcap (x \sqcup y) = x \sqcap x$	(5b) $x \sqcup (x \sqcap y) = x \sqcup x$
(6a) $x \sqcap (y \vee z) = (x \sqcap y) \vee (x \sqcap z)$	(6b) $x \sqcup (y \wedge z) = (x \sqcup y) \wedge (x \sqcup z)$
(7a) $x \sqcap (x \vee y) = x \sqcap x$	(7b) $x \sqcup (x \wedge y) = x \sqcup x$
(8a) $\neg\neg(x \sqcap y) = x \sqcap y$	(8b) $\lrcorner\lrcorner(x \sqcup y) = x \sqcup y$
(9a) $x \sqcap \neg x = \bot$	(9b) $x \sqcup \lrcorner x = \top$
(10a) $\neg\bot = \top \sqcap \top$	(10b) $\lrcorner\top = \bot \sqcup \bot$
(11a) $\neg\top = \bot$	(11b) $\lrcorner\bot = \top$

12 $(x \sqcap x) \sqcup (x \sqcap x) = (x \sqcup x) \sqcap (x \sqcup x)$,

where $x \vee y := \neg(\neg x \sqcap \neg y)$, and $x \wedge y := \lrcorner(\lrcorner x \sqcup \lrcorner y)$.

Definition 4. A dBa $\mathbf{D}$ is called *pure* if for all $x \in D$, either $x \sqcap x = x$ or $x \sqcup x = x$.

Theorem 1. [14] Let $\mathbb{K}$ be a context.

1. $\underline{\mathfrak{P}}(\mathbb{K})$ forms a dBa.
2. $\underline{\mathfrak{H}}(\mathbb{K})$ forms a pure dBa.

In the following, let $\mathbf{D} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ be a dBa.

On abstraction of the partial order $\leq$ in the protoconcept algebra $\underline{\mathfrak{P}}(\mathbb{K})$, a relation $\sqsubseteq$ is defined on $\mathbf{D}$ as follows. For any $x, y \in D$,

$$x \sqsubseteq y \text{ if and only if } x \sqcap y = x \sqcap x \text{ and } x \sqcup y = y \sqcup y.$$

$\sqsubseteq$ is shown to be a quasi-order on $\mathbf{D}$ [5, 14].

Notation 3 For $x \in D$, $x_\sqcap := x \sqcap x$ and $x_\sqcup := x \sqcup x$. $D_\sqcap := \{x \in D \mid x_\sqcap = x\}$ and $D_\sqcup := \{x \in D \mid x_\sqcup = x\}$.

Proposition 1. [9]
(i) $\mathbf{D}_\sqcap := (D_\sqcap, \sqcap, \vee, \neg, \bot, \neg\bot)$ is a Boolean algebra whose order relation is the restriction of $\sqsubseteq$ to $D_\sqcap$ and is denoted by $\sqsubseteq_\sqcap$.
(ii) $\mathbf{D}_\sqcup := (D_\sqcup, \sqcup, \wedge, \lrcorner, \top, \lrcorner\top)$ is a Boolean algebra whose order relation is the restriction of $\sqsubseteq$ to $D_\sqcup$ and it is denoted by $\sqsubseteq_\sqcup$.
(iii) For any $x, y \in D$, $x \sqsubseteq y$ if and only if $x \sqcap x \sqsubseteq y \sqcap y$ and $x \sqcup x \sqsubseteq y \sqcup y$, that is $x_\sqcap \sqsubseteq_\sqcap y_\sqcap$ and $x_\sqcup \sqsubseteq_\sqcup y_\sqcup$.

Proposition 2. [5] Let $x, y, a \in D$. The following hold.

1. $x \sqcap \bot = \bot$ and $x \sqcup \bot = x \sqcup x$, that is $\bot \sqsubseteq x$.
2. $x \sqcup \top = \top$ and $x \sqcap \top = x \sqcap x$, that is $x \sqsubseteq \top$.
3. $x \sqsubseteq y$ and $y \sqsubseteq x$ if and only if $x \sqcap x = y \sqcap y$ and $x \sqcup x = y \sqcup y$.
4. $x \sqcap y \sqsubseteq x, y \sqsubseteq x \sqcup y$.
5. $x \sqsubseteq y$ implies $x \sqcap a \sqsubseteq y \sqcap a$ and $x \sqcup a \sqsubseteq y \sqcup a$.

3 An algebraic investigation of double Boolean algebras

In [5], Kwuida showed that for any $x, y \in \mathfrak{P}(\mathbb{K})$, $x \sqcap x \leq (x \sqcap y) \sqcup (x \sqcap \neg y)$ and $(x \sqcup y) \sqcap (x \sqcup \lrcorner y) \leq x \sqcup x$. He next redefined Wille's dBa by adding two new axioms to Definition 3 (Section 2):
(a) $x \sqcap x \sqsubseteq (x \sqcap y) \sqcup (x \sqcap \neg y)$,
(b) $(x \sqcup y) \sqcap (x \sqcup \lrcorner y) \sqsubseteq x \sqcup x$.

The name 'double Boolean algebra' was retained for the new class of algebras by Kwuida. In this section, we prove a sequence of results and conclude in Corollary 1 that the two axioms (a)-(b) above are derivable from Definition 3. In the following, let $\boldsymbol{D} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ denote Wille's dBa.

Proposition 3. For any $x, y \in D$, the following hold.

1. $\neg x \sqcap \neg x = \neg x$ and $\lrcorner x \sqcup \lrcorner x = \lrcorner x$, that is $\neg x = (\neg x)_\sqcap \in D_\sqcap$ and $\lrcorner x = (\lrcorner x)_\sqcup \in D_\sqcup$.
2. $x \sqsubseteq y$ if and only if $\neg y \sqsubseteq \neg x$ and $\lrcorner y \sqsubseteq \lrcorner x$.
3. $\neg\neg x = x \sqcap x$ and $\lrcorner\lrcorner x = x \sqcup x$.
4. $x \vee y \in D_\sqcap, x \wedge y \in D_\sqcup$.
5. $\neg(x \vee y) = \neg x \sqcap \neg y$ and $\neg(x \sqcap y) = \neg x \vee \neg y$.
6. $\lrcorner(x \wedge y) = \lrcorner x \sqcup \lrcorner y$ and $\lrcorner(x \sqcup y) = \lrcorner x \wedge \lrcorner y$.

Proof. 1. Let $x \in D$. Axiom (1a) gives $x \sqcap x \in D_\sqcap$. Axiom (4a) and Proposition 1(i) give $\neg x = \neg(x \sqcap x) \in D_\sqcap$. The other part is proved dually.
2. Let $x, y \in D$ then $x \sqsubseteq y$ if and only if $x_\sqcap \sqsubseteq_\sqcap y_\sqcap$ and $x_\sqcup \sqsubseteq_\sqcup y_\sqcup$, by (iii) of Proposition 1. $x_\sqcap \sqsubseteq_\sqcap y_\sqcap$ and $x_\sqcup \sqsubseteq_\sqcup y_\sqcup$ if and only if $\neg y_\sqcap \sqsubseteq_\sqcap \neg x_\sqcap$ and $\lrcorner y_\sqcup \sqsubseteq_\sqcup \lrcorner x_\sqcup$, by (i) and (ii) of Proposition 1. $\neg y_\sqcap \sqsubseteq_\sqcap \neg x_\sqcap$ and $\lrcorner y_\sqcup \sqsubseteq_\sqcup \lrcorner x_\sqcup$ if and only if $\neg y \sqsubseteq_\sqcap \neg x$ and $\lrcorner y \sqsubseteq_\sqcup \lrcorner x$, by axioms (4a) and (4b). Finally, $\neg y \sqsubseteq_\sqcap \neg x$ and

$\lrcorner y \sqsubseteq_{\sqcup} \lrcorner x$ if and only if $\neg y \sqsubseteq \neg x$ and $\lrcorner y \sqsubseteq \lrcorner x$, by Proposition 1 and part (1) of this proposition.
3. Proof follows from axioms ($4a$) and ($8a$).
4. By part (1) of this proposition, $\neg x \in D_{\sqcap}$ and $\lrcorner x \in D_{\sqcup}$. Using the definitions of $\wedge$ and $\vee$ and closure with respect to the operations in $D_{\sqcap}$ and $D_{\sqcup}$, we get the result.
5. $\neg(x \vee y) = \neg(\neg(\neg x \sqcap \neg y)) = \neg x \sqcap \neg y$, by axiom ($8a$).
Now $\neg(x \sqcap y) = \neg((x \sqcap x) \sqcap (y \sqcap y))$(by axioms ($1a$) – ($3a$))
$= \neg(\neg\neg x \sqcap \neg\neg y)$ (3 of Proposition 3)
$= \neg x \vee \neg y$ (by definition of $\vee$).
6. Proof is dual to the proof of 5. □

Proposition 4. For any $x, y \in D$, we have the following.

1. $x \sqsubseteq \lrcorner y$ if and only if $y \sqsubseteq \lrcorner x$.
2. $\neg x \sqsubseteq y$ if and only if $\neg y \sqsubseteq x$.

Proof. Follows from Proposition 3 and the facts that $\neg\neg x \sqsubseteq x$ and $x \sqsubseteq \lrcorner\lrcorner x$ for any $x \in D$. □

Theorem 2. For all $x, y \in D$, the following hold.

1. $x \sqcap \neg(x \sqcup y) = \bot$
2. $\neg(x \sqcup y) = \neg(x \sqcup y) \sqcap \neg x$
3. $x \sqcap y = x \sqcap \neg(x \sqcap \neg y)$
4. $x \sqcup (y \sqcap \neg x) = x \sqcup (y \sqcap y)$
5 $(x \sqcap y) \sqcup (x \sqcap \neg y) = (x \sqcap x) \sqcup (x \sqcap x)$
6. $x \sqcup \lrcorner(x \sqcap y) = \top$
7. $\lrcorner(x \sqcap y) = \lrcorner(x \sqcap y) \sqcup \lrcorner x$
8. $x \sqcup y = x \sqcup \lrcorner(x \sqcup \lrcorner y)$
9. $x \sqcap (y \sqcup \lrcorner x) = x \sqcap (y \sqcup y)$
10 $(x \sqcup y) \sqcap (x \sqcup \lrcorner y) = (x \sqcup x) \sqcap (x \sqcup x)$

Proof. We give the proofs for $1, 2, 3, 4, 5$. The properties of commutativity and associativity are used without mention in the proofs. Let $x, y \in D$.
Proof of 1: $\bot = x \sqcap \bot$ (by Proposition 2)
$= x \sqcap ((x \sqcup y) \sqcap \neg(x \sqcup y))$ (by axiom (9a))
$= (x \sqcap (x \sqcup y)) \sqcap \neg(x \sqcup y)$
$= (x \sqcap x) \sqcap \neg(x \sqcup y)$ (by axiom (5a))
$= x \sqcap \neg(x \sqcup y)$ (by axiom (1a)).

Proof of 2: $\neg(x \sqcup y) \sqcap \neg x = \neg(x \sqcup y) \sqcap \neg(x \sqcap x)$ (by axiom (4a))
$= \neg(x \sqcup y) \sqcap \neg((x \sqcup y) \sqcap x)$ (by axioms (5a))
$= \neg(x \sqcup y) \sqcap \neg(x \sqcup y)$ (as $\neg(x \sqcup y) \sqsubseteq \neg((x \sqcup y) \sqcap x)$)
$= \neg(x \sqcup y)$ (by (1) of Proposition 3).

Proof of 3: $x \sqcap \neg(x \sqcap \neg y) = x \sqcap (\neg x \vee \neg\neg y)$ (by 5 of Proposition 3)
$= x \sqcap (\neg x \vee (y \sqcap y))$ (by axioms (8a) and (4a))
$= (x \sqcap \neg x) \vee (x \sqcap (y \sqcap y))$ (by axiom (6a))
$= \bot \vee (x \sqcap y)$ (by axioms (9a) and (1a))
$= x \sqcap y$ (as $x \sqcap y \in D_{\sqcap}$).

Proof of 4: To prove 4, first we show that $y \sqcap \top = y \sqcap (x \sqcup (y \sqcap \neg x))$.
Now $y \sqcap \top = y \sqcap \top \sqcap \top$ (by axiom (1a))
$= y \sqcap \neg\bot$ (by axiom (10a))
$= y \sqcap \neg((y \sqcap \neg x) \sqcap \neg(x \sqcup (y \sqcap \neg x)))$ (by 1 of this proposition)
$= y \sqcap \neg(y \sqcap (\neg x \sqcap \neg(x \sqcup (y \sqcap \neg x))))$
$= y \sqcap \neg(y \sqcap \neg(x \sqcup (y \sqcap \neg x)))$ (by 2 of this proposition)
$= y \sqcap (x \sqcup (y \sqcap \neg x))$ (by 3 of this proposition). (*)
Now $x \sqcup (y \sqcap y) = x \sqcup (y \sqcap y) \sqcup (y \sqcap y)$ (by axioms (1b))
$= x \sqcup ((y \sqcap y) \sqcup ((y \sqcap y) \sqcap \neg x))$ (by axiom (5b))
$= x \sqcup ((y \sqcap y) \sqcup (y \sqcap \neg x))$ (by (1a)) $= x \sqcup ((y \sqcap \top) \sqcup (y \sqcap \neg x))$ (by Proposition 2)
$= x \sqcup (y \sqcap (x \sqcup (y \sqcap \neg x)) \sqcup (y \sqcap \neg x)$ (by (*))
$= (x \sqcup (y \sqcap \neg x)) \sqcup (y \sqcap (x \sqcup (y \sqcap \neg x)))$
$= (x \sqcup (y \sqcap \neg x)) \sqcup (x \sqcup (y \sqcap \neg x))$(by axiom(5b))
$= (x \sqcup (y \sqcap \neg x))$ (by axiom (1b)).

Proof of 5: $(x \sqcap y) \sqcup (x \sqcap \neg y) = (x \sqcap y) \sqcup (x \sqcap \neg(x \sqcap \neg\neg y))$ (by 3 of this proposition)
$= (x \sqcap y) \sqcup (x \sqcap \neg(x \sqcap y \sqcap y))$ (by axioms (4a) and (8a))
$= (x \sqcap y) \sqcup (x \sqcap \neg(x \sqcap y))$ (by axiom (1a))
$= (x \sqcap y) \sqcup (x \sqcap x)$ (by 4 of this proposition)
$= ((x \sqcap x) \sqcap y) \sqcup (x \sqcap x)$ (by axiom (1a))
$= (x \sqcap x) \sqcup (x \sqcap x)$ (by axiom (5b)).

The proofs of $6, 7, 8, 9, 10$ are dual to those of $1, 2, 3, 4, 5$ respectively. □

We now conclude that the new structure defined by Kwuida is, in fact, equivalent to Wille's dBa:

Corollary 1. For all $x, y \in D$, the following hold.

(a) $(x \sqcup y) \sqcap (x \sqcup \lrcorner y) \sqsubseteq x \sqcup x$.
(b) $x \sqcap x \sqsubseteq (x \sqcap y) \sqcup (x \sqcap \neg y)$.

Proof. Follows from 5 and 10 of Theorem 2, and Proposition 2(4). □

Kwuida generalized the notion of prime filter (ideal) of Boolean algebras to that of 'primary filter' ('primary ideal') for dBas defined in [5], and proved the prime ideal theorem. Corollary 1 enables us to extend these notions and the theorem to Wille's dBas. Let $\mathbf{D} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ be a dBa.

Definition 5. A subset F of D is a *filter* in $\mathbf{D}$ if and only if $x \sqcap y \in F$ for all $x, y \in F$, and for all $z \in D$ and $x \in F, x \sqsubseteq z$ implies that $z \in F$. An *ideal* in a dBa is defined dually.
A filter F (ideal I) is *proper* if and only if $F \neq D$ ($I \neq D$). A proper filter F (ideal I) is called *primary* if and only if $x \in F$ or $\neg x \in F$ ($x \in I$ or $\lrcorner x \in I$), for all $x \in D$.
The set of primary filters is denoted by $\mathcal{F}_{pr}(\mathbf{D})$; the set of all primary ideals is denoted by $\mathcal{I}_{pr}(\mathbf{D})$.

Theorem 3 (Prime ideal theorem for dBas). Let $\mathbf{D}$ be a dBa, F a filter in $\mathbf{D}$ and I an ideal in $\mathbf{D}$ such that $F \cap I = \emptyset$. There exists a primary filter G and a primary ideal J of $\mathbf{D}$ such that $F \subseteq G,\ I \subseteq J$ and $G \cap J = \emptyset$.

4 Representation theorems for double Boolean algebras: a discussion

The dBa is obtained by abstraction of certain properties of the protoconcept algebra and the pure dBa is obtained from the semiconcept algebra. Now two fundamental questions raised are as follows:
Q1: Is every dBa (pure dBa) $\mathbf{D}$ isomorphic to the protoconcept (semiconcept) algebra of some context?
Q2: Can every dBa (pure dBa) $\mathbf{D}$ be embedded into the protoconcept (semiconcept) algebra of some context?

The answer of Q2 is yes. Some notations and definitions [14] required to state the results related to Q2 are as follows.

Notation 4

- $\mathcal{F}_p(\mathbf{D}) := \{F \subseteq D | F$ is a filter of $\mathbf{D}$ and $F \cap D_\sqcap$ is a prime filter in $\mathbf{D}_\sqcap\}$.
- $\mathcal{I}_p(\mathbf{D}) := \{I \subseteq D | I$ is an ideal of $\mathbf{D}$ and $I \cap D_\sqcup$ is a prime ideal in $\mathbf{D}_\sqcup\}$.
- For any $x \in D$, $F_x := \{F \in \mathcal{F}_p(\mathbf{D}) \mid x \in F\}$ and $I_x := \{I \in \mathcal{I}_p(\mathbf{D}) \mid x \in I\}$.

Definition 6. Let $\mathbf{D}$ and $\mathbf{M}$ be two dBas. A map $h : M \to D$ is called a *homomorphism* if h preserves the operations in the algebras.
h is called *quasi-injective*, when $x \sqsubseteq y$ if and only if $h(x) \sqsubseteq h(y)$, for all $x, y \in M$. A quasi-injective and surjective homomorphism is called a *quasi-isomorphism* and a bijective homomorphism is called an *isomorphism*.

In [14], for every dBa $\mathbf{D}$, Wille defined a context $\mathbb{K}(\mathbf{D}) := (\mathcal{F}_p(\mathbf{D}), \mathcal{I}_p(\mathbf{D}), \Delta)$, where for all $F \in \mathcal{F}_p(\mathbf{D})$ and for all $I \in \mathcal{I}_p(\mathbf{D})$, $F \Delta I$ if and only if $F \cap I \neq \emptyset$. The following theorem was proved.

Theorem 4. [14] The map $i : \mathbf{D} \to \underline{\mathfrak{P}}(\mathbb{K}(\mathbf{D}))$ defined by $i(a) := (F_a, I_a)$ for all $a \in \mathbf{D}$, is a quasi-injective homomorphism.

Balbiani subsequently showed the following for a pure dBa.

Theorem 5. [1] Let $\mathbf{D}$ be a pure dBa. Then the map $i : \mathbf{D} \to \underline{\mathfrak{H}}(\mathbb{K}(\mathbf{D}))$ defined by $i(a) := (F_a, I_a)$ for all $a \in \mathbf{D}$, is an injective homomorphism.

For addressing Q1, we need some definitions.

Definition 7. [10] Let $\mathbf{D}$ be a dBa.

1. $\mathbf{D}$ is *complete* if and only if the Boolean algebras $\mathbf{D}_\sqcup$ and $\mathbf{D}_\sqcap$ are complete.
2. It is *contextual* if and only if the quasi-order $\sqsubseteq$ on $\mathbf{D}$ is a partial order.
3. It is *fully contextual* if and only if for each $y \in D_\sqcap$ and $x \in D_\sqcup$ with $y_\sqcup = x_\sqcap$, there is a unique $z \in D$ with $z_\sqcap = x$ and $z_\sqcup = y$.

In [10], Vormbrock showed that any complete pure dBa $\mathbf{D}$ whose Boolean algebras $\mathbf{D}_\sqcap$ and $\mathbf{D}_\sqcup$ are atomic, is isomorphic to the semiconcept algebra of some context. On the other hand, any complete fully contextual dBa whose Boolean algebras $\mathbf{D}_\sqcap$ and $\mathbf{D}_\sqcup$ are atomic, is isomorphic to the protoconcept algebra of some context. Now note that not all dBas are complete: consider Boolean algebras that are not complete. As proved in Theorem 4, any dBa is quasi-isomorphic to a subalgebra of the protoconcept algebra of some context. A question then is to characterise this subalgebra. In [2], Breckner and Săcărea address this question for the class of 'regular' dBas, which are just contextual dBas mentioned in Definition 7. We now discuss their results along with related definitions.

4.1 Regular dBas

In the following, let $\mathbb{K} := (G, M, R)$ be a context. Let us first note that the dBa $\underline{\mathfrak{P}}(\mathbb{K})$ formed by the protoconcepts of $\mathbb{K}$ is regular. Topologies are now introduced into the picture. Recall that for a topological space (X, τ), a subset A of X is said to be *clopen* if it is both closed and open in (X, τ).

Definition 8. Let τ be a topology on G and ρ a topology on M. Then $\mathbb{K}^{DB} := ((G, \tau), (M, \rho), R)$ is called a *context on topological spaces.*
A *clopen protoconcept* (A, B) of $\mathbb{K}^{DB}$ is a protoconcept of $\mathbb{K}$ such that A and B are clopen in the topological spaces (G, τ), (M, ρ) respectively.

Notation 5 The set of all clopen protoconcepts of $\mathbb{K}^{DB}$ is denoted by $\mathfrak{P}^{co}(\mathbb{K}^{DB})$.

Definition 9. [2] A context on topological spaces $\mathbb{K}^{DB} := ((G, \tau), (M, \rho), R)$ is called a *DB-topological context* if and only if the following are satisfied.

1. For every clopen subset A of (G, τ), A' is clopen in (M, ρ); for every clopen subset B of (M, ρ), B' is clopen in (G, τ).
2. The extents of all clopen protoconcepts of $\mathbb{K}^{DB}$ give a subbasis for the closed and open sets in (G, τ), while the intents of all clopen protoconcepts of $\mathbb{K}^{DB}$ give a subbasis for the closed and open sets in (M, ρ).

In [2], the following propositions are proved.

Proposition 5. For a DB-topological context $\mathbb{K}^{DB}$, the set $\mathfrak{P}^{co}(\mathbb{K}^{DB})$ of all clopen protoconcepts of $\mathbb{K}^{DB}$ forms a subalgebra of the regular dBa $\underline{\mathfrak{P}}(\mathbb{K})$.

This subalgebra is denoted by $\underline{\mathfrak{P}}^{co}(\mathbb{K}^{DB})$.
For a regular dBa $\mathbf{D}$ and the context $\mathbb{K}(\mathbf{D}) := (\mathcal{F}_p(\mathbf{D}), \mathcal{I}_p(\mathbf{D}), \Delta)$ mentioned earlier, define a topology τ on $\mathcal{F}_p(\mathbf{D})$ with a subbasis for the closed sets given by $\{F_x : x \in D\}$, and a topology ρ on $\mathcal{I}_p(\mathbf{D})$ with a subbasis for the closed sets given by $\{I_x : x \in D\}$. So one obtains the context on topological spaces $\mathbb{K}^{DB}(\mathbf{D}) := ((\mathcal{F}_p(\mathbf{D}), \tau), (\mathcal{I}_p(\mathbf{D}), \rho), \Delta)$.

Proposition 6. $\mathbb{K}^{DB}(\mathbf{D}) := ((\mathcal{F}_p(\mathbf{D}), \tau), (\mathcal{I}_p(\mathbf{D}), \rho), \Delta)$ is a DB-topological context.

The following theorem is then presented in [2].

Theorem 6. Let $\mathbf{D}$ be a regular dBa. The map

$$i : \mathbf{D} \to \underline{\mathfrak{P}}^{co}(\mathbb{K}^{DB}(\mathbf{D})) \text{ such that } i(a) := (F_a, I_a) \text{ for any } a \in D,$$

is an isomorphism.

From Theorem 4 it follows that the map i is a quasi-injective homomorphism. In particular when $\mathbf{D}$ is regular, i becomes an embedding. Theorem 6 claims that i is moreover surjective onto $\underline{\mathfrak{P}}^{co}(\mathbb{K}^{DB}(\mathbf{D}))$. In Example 1 below, we show that this need not be the case.

A counterexample to Theorem 6: To establish that the map i defined in Theorem 6 is surjective, it is shown that the extent of any clopen protoconcept is of the form F_a for some $a \in D$. The intent would also be of the form I_b for some $b \in D$. An observation is that any pair (F_a, I_b) is a protoconcept ($F_a'' = I_b'$), if and only if $F_{a \sqcup a} = F_{b \sqcup b}$ (equivalently, $I_{a \sqcap a} = I_{b \sqcap b}$). Now if we choose $a \neq b$ such that $F_{a \sqcup a} = F_{b \sqcup b}$ then (F_a, I_b) is a clopen protoconcept of $\mathbb{K}^{DB}(\mathbf{D})$, but there is no guarantee that there exists a $c \in D$ such that $i(c) = (F_a, I_b)$. This is what we verify in Example 1 below. Before doing so, we observe that primary filters (ideals) of a dBa $\mathbf{D}$ introduced by Kwuida are exactly the extensions of prime filters (ideals) of the Boolean algebra $\mathbf{D}_{\sqcup}(\mathbf{D}_{\sqcap})$:

Proposition 7. For a double Boolean algebra $\mathbf{D}$,

1. $\mathcal{F}_{pr}(\mathbf{D}) = \mathcal{F}_p(\mathbf{D})$.
2. $\mathcal{I}_{pr}(\mathbf{D}) = \mathcal{I}_p(\mathbf{D})$.

Proof. The proof of 2 is dual to the proof of 1. We prove 1. Let $F \in \mathcal{F}_p(\mathbf{D})$. $F \cap D_{\sqcap}$ is then a prime filter in $\mathbf{D}_{\sqcap}$. Let $x \in \mathbf{D}$ and $x \notin F$. Then $x \sqcap x \notin F$ (otherwise $x \in F$) and hence $x \sqcap x \notin F \cap D_{\sqcap}$. As $F \cap D_{\sqcap}$ is a prime filter in $\mathbf{D}_{\sqcap}$, using axiom (4a), $\neg x = \neg(x \sqcap x) \in F \cap D_{\sqcap}$, giving $\neg x \in F$. Hence $F \in \mathcal{F}_{pr}(\mathbf{D})$. For the converse, let us assume that $F \in \mathcal{F}_{pr}(\mathbf{D})$. Let $x \in D_{\sqcap} (\subseteq D)$. Then $x \in F$ or $\neg x \in F$, as F is a primary filter in $\mathbf{D}$. Therefore we have $x \in F \cap D_{\sqcap}$ or $\neg x \in F \cap D_{\sqcap}$, and hence $F \cap D_{\sqcap}$ is a prime filter in $\mathbf{D}_{\sqcap}$. So $F \in \mathcal{F}_p(\mathbf{D})$. □

Corollary 2. For a double Boolean algebra $\mathbf{D}$, there is one-one and onto correspondence between the set of primary filters (ideals) of $\mathbf{D}$ and the set of prime filters (ideals) of $\mathbf{D}_{\sqcap}(\mathbf{D}_{\sqcup})$.

Proof. Let F be a primary filter of $\mathbf{D}$. Then by Proposition 7, $F \cap D_{\sqcap}$ is a prime filter of $\mathbf{D}_{\sqcap}$. If F_0 is a prime filter in $\mathbf{D}_{\sqcap}$, it can be shown that $F = \{y : x \sqsubseteq y \text{ for some } x \in F_0\}$ is a filter in $\mathbf{D}$ such that $F \cap \mathbf{D}_{\sqcap} = F_0$. So by Proposition 7, F is a primary filter in $\mathbf{D}$. The case for ideals is done dually. □

Lemma 1. [14] For all $x \in \mathbf{D}$, $F_x' = I_{x_{\sqcap\sqcup}}$ and $I_x' = F_{x_{\sqcup\sqcap}}$.

Example 1. For a regular dBa $\mathbf{D}$, the map i defined in Theorem 6 may not be surjective. This is established by the following example of a regular dBa $\mathbf{D}$ [5]. Consider the three element chain $D := \{\bot, a, \top\}$, where $\top \sqcap \top = a = \bot \sqcup \bot$ and $a \sqcap a = a \sqcup a = a$. For all $x \in D$, $\neg x = \bot$ and $\lrcorner x = \top$. The Boolean algebras of the regular dBa are $\mathbf{D}_{\sqcap} = (\{a, \bot\}, \sqcap, \wedge, \neg, \bot, \neg\bot)$ and $\mathbf{D}_{\sqcup} = (\{a, \top\}, \sqcup, \vee, \lrcorner, \top, \lrcorner\top)$. Then by Corollary 2, we have $\mathcal{F}_p(\mathbf{D}) = \{\{a, \top\}\}$, as $\{a\}$ is the only prime filter in the Boolean algebra $\mathbf{D}_{\sqcap}$. Similarly, $\mathcal{I}_p(\mathbf{D}) = \{\{a, \bot\}\}$ as $\{a\}$ is the only prime ideal in the Boolean algebra $\mathbf{D}_{\sqcup}$. Therefore $\mathbb{K}^{DB}(\mathbf{D}) = ((\{\{a, \top\}\}, \{F_a, F_\top, F_\bot\}), (\{\{a, \bot\}\}, \{I_a, I_\top, I_\bot\}), \Delta)$, where $\Delta = \mathcal{F}_p(\mathbf{D}) \times \mathcal{I}_p(\mathbf{D})$. The image elements under i are: $i(a) = (\mathcal{F}_p(\mathbf{D}), \mathcal{I}_p(\mathbf{D}))$, $i(\top) = (\mathcal{F}_p(\mathbf{D}), \emptyset)$, $i(\bot) = (\emptyset, \mathcal{I}_p(\mathbf{D}))$. Then the pair $x := (\emptyset, \emptyset) = (F_\bot, I_\top)$ is a clopen protoconcept of $\mathbb{K}^{DB}(\mathbf{D})$. But the element x has no pre-image under the map i, and hence the map i is not surjective.

The special case for Boolean algebras: We know that the class of Boolean algebras is a subclass of that of regular dBas. For this subclass however, the isomorphism result (Theorem 6) holds, as we note in Theorem 7 below.

Note that in the case of a Boolean algebra $\mathbf{B}$, the topological spaces $(\mathcal{F}_p(\mathbf{B}), \tau)$ and $(\mathcal{I}_p(\mathbf{B}), \rho)$ are homeomorphic Stone spaces. Moreover, we have

Lemma 2. For all $a \in \mathbf{B}$, (F_a, I_a) is a clopen concept of the DB-topological context $\mathbb{K}^{DB}(\mathbf{B}) := ((\mathcal{F}_p(\mathbf{B}), \tau), (\mathcal{I}_p(\mathbf{B}), \rho), \Delta)$.

Proof. Since $\mathbf{B}$ is a Boolean algebra, for all $a \in \mathbf{B}$ we have that F_a is clopen in $(\mathcal{F}_p(\mathbf{B}), \tau)$ and I_a is clopen in $(\mathcal{I}_p(\mathbf{B}), \rho)$. Now from Lemma 1 it follows that $F'_a = I_{a_{\sqcap\sqcup}} = I_a$, as $a_{\sqcap\sqcup} = a$. Similarly, we can show that $I'_a = F_a$. So (F_a, I_a) is a clopen concept of the DB-topological context $\mathbb{K}^{DB}(\mathbf{B})$. □

Theorem 7. Let $\mathbf{B} := (B, \sqcup, \sqcap, ^c, \top, \bot)$ be a Boolean algebra. Then $\mathbf{B}$ is isomorphic to $\underline{\mathfrak{P}}^{co}(\mathbb{K}^{DB}(\mathbf{B}))$.

Proof. One shows that the map $i : \mathbf{B} \to \underline{\mathfrak{P}}^{co}(\mathbb{K}^{DB}(\mathbf{B}))$ defined by $i(a) := (F_a, I_a)$ for any $a \in B$, is a Boolean algebra isomorphism.
Let $(X, Y) \in \mathfrak{P}^{co}(\mathbb{K}^{DB}(\mathbf{B}))$. Then X is clopen in $(\mathcal{F}_p(\mathbf{B}), \tau)$ and Y is clopen in $(\mathcal{I}_p(\mathbf{B}), \rho)$. Since $\mathbf{B}$ is a Boolean algebra, every clopen set in $(\mathcal{F}_p(\mathbf{B}), \tau)$ is of the form F_a for some $a \in B$ and every clopen set in $(\mathcal{I}_p(\mathbf{B}), \rho)$ is of the form I_b for some $b \in B$. So $X = F_{a^0}$ for some $a^0 \in B$ and $Y = I_{b^0}$ for some $b^0 \in B$. Since (X, Y) is a protoconcept, $F''_{a^0} = I'_{b^0}$ and so $F_{a^0} = F_{b^0}$. Now, if possible, let us suppose that $a^0 \neq b^0$. This would imply that either $a^0 \nleq b^0$ or $b^0 \nleq a^0$. Let $a^0 \nleq b^0$. By the prime ideal theorem of Boolean algebras, there exists a prime filter F such that $a^0 \in F$ and $b^0 \notin F$ – which is not possible, as $F_{a^0} = F_{b^0}$. In case $b^0 \nleq a^0$, one can similarly show that $F_{a^0} \neq F_{b^0}$. So $a^0 = b^0$. Therefore by Lemma 2, it follows that $\mathfrak{P}^{co}(\mathbb{K}^{DB}(\mathbf{B}))$ is a collection of clopen concepts of $\mathbb{K}^{DB}(\mathbf{B})$ and since $\mathfrak{P}^{co}(\mathbb{K}^{DB}(\mathbf{B}))$ is closed under $\sqcup, \sqcap$, $(\mathfrak{P}^{co}(\mathbb{K}^{DB}(\mathbf{B})), \sqcup, \sqcap)$ forms a lattice. Now for all clopen protoconcepts $(F_a, I_b) \in \mathfrak{P}^{co}(\mathbb{K}^{DB}(\mathbf{B}))$, $\neg(F_a, I_a) = (\mathcal{F}_p(\mathbf{B}) \setminus F_a, (\mathcal{F}_p(\mathbf{B}) \setminus F_a)') = (F_{a^c}, I_{a^c}) = ((\mathcal{I}_p(\mathbf{B}) \setminus$

$I_a)', \mathcal{I}_p(\mathbf{B}) \setminus I_a) = \lrcorner(F_a, I_a)$ and since $\top := (\mathcal{F}_p(\mathbf{B}), \emptyset)$, $\bot := (\emptyset, \mathcal{I}_p(\mathbf{B}))$ belongs to $\mathfrak{P}^{co}(\mathbb{K}^{DB}(\mathbf{B}))$ then $\underline{\mathfrak{P}}^{co}(\mathbb{K}^{DB}(\mathbf{B})) = (\mathfrak{P}^{co}(\mathbb{K}^{DB}(\mathbf{B})), \sqcup, \sqcap, \neg, \top, \bot)$ forms a bounded complemented lattice. Considering the structures as dBas, Theorem 4 gives that the map i is a dBa homomorphism. Injectivity and surjectivity of the map i follow from the above proof also. Hence i becomes a Boolean algebra isomorphism. □

5 Conclusion

In this work, we show that the new class of algebras defined in [5] by Kwuida is the same as the class of double Boolean algebras defined by Wille. Hence the prime ideal theorem proved in [5] holds for Wille's dBas. We next briefly survey representation results obtained for different kinds of dBas. In particular, it is observed through a counterexample that the representation theorem proved in [2] for regular dBas may not hold. It is shown that the theorem is true however, for the special case of Boolean algebras.

From Theorem 4 and the work of Breckner and Săcărea it follows that every dBa $\mathbf{D}$ is quasi-isomorphic to some subalgebra of $\underline{\mathfrak{P}}^{co}(\mathbb{K}^{DB}(\mathbf{D}))$. Moreover, a regular dBa $\mathbf{D}$ is isomorphic to some subalgebra of $\underline{\mathfrak{P}}^{co}(\mathbb{K}^{DB}(\mathbf{D}))$. Characterization of this subalgebra remains an open question.

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