Notes on Integer Partitions

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Abstract. Some observations concerning the lattices of integer partitions are presented. We determine the size of the standard contexts, discuss a recursive construction and show that the lattices have unbounded breadth.

1 Introduction

Integer partitions have been studied since the time of Leibnitz and Euler and are still of interest (see e.g. Knuth [8] for a contemporary contribution and Andrews & Eriksson [2] for a monography).

The set of all partitions of n, when endowed with the "dominance order", forms a complete lattice. Almost 50 years ago Thomas Brylawski published a first major investigation on this topic. Building on his results we describe these lattices as concept lattices and extend some of his findings. Our presentation follows Brylawski's paper [3], where also the proofs not given here can be found.

We suggest that the reader, before going through the technical definitions below, takes a look at Figure 1. It shows the lattice of the 42 partitions of the integer 10, ordered by dominance.

2 Basic notions and facts

A **partition** of a positive integer n is a nonincreasing n-tuple $\underline{a} := (a_1, \ldots, a_n)$ of integers $a_i \geq 0$ such that $\sum_{i=1}^{n} a_i = n$. Usually we write such a partition as

$$n = a_1 + a_2 + \ldots,$$

omitting summands which are equal to zero. The set of all partitions of n is denoted $\text{Part}(n)$.

A partition $\underline{a} := (a_1, \ldots, a_n)$ **dominates** a partition $\underline{b} := (b_1, \ldots, b_n)$ iff

$$\sum_{j=1}^{i} a_j \;\geq\; \sum_{j=1}^{i} b_j \qquad \text{holds for all } i.$$

We mention an immediate consequence of this definition. Let $l(\underline{a})$ denote the **length**, i.e., the number of nonzero summands of the partition $\underline{a}$. When $\underline{a} \geq \underline{b}$, then $\sum_{j=1}^{l(\underline{b})} a_j \geq \sum_{j=1}^{l(\underline{b})} b_j = n$, which implies $l(\underline{a}) \leq l(\underline{b})$. Thus longer partitions tend to be the lower part of the order.

ISBN 978–9949–83–557–7, Tallinn University of Technology, 2020.

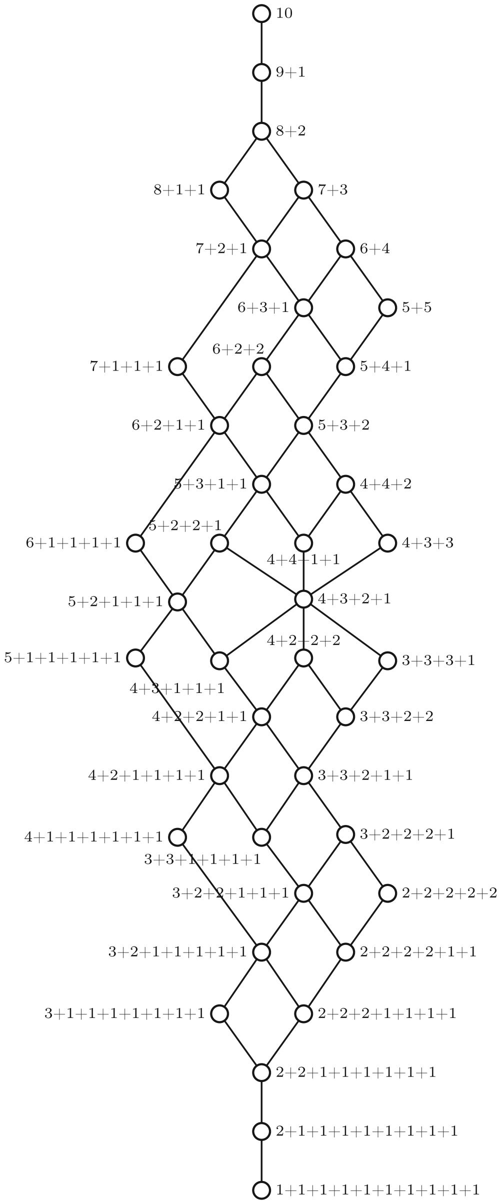

Fig. 1. The lattice of all partitions of the integer 10 (from [4]).

To every partition $\underline{a}$ of n corresponds a unique **associated** $(n+1)$**-tuple** $\widehat{\underline{a}} := (\widehat{a}_0, \dots, \widehat{a}_n)$, defined by

$$\widehat{a}_i := \sum_{j=1}^{i} a_j \qquad \text{for all } i \in \{0, \dots n\}.$$

These associated tuples can be characterized as being precisely the nondecreasing (i.e., $\widehat{a}_i \geq \widehat{a}_{i-1}$) and concave (i.e., $2\widehat{a}_i \geq \widehat{a}_{i-1} + \widehat{a}_{i+1}$) $(n+1)$-tuples of integers with $\widehat{a}_0 = 0$ and $\widehat{a}_n = n$.

The dominance order defined above corresponds to the componentwise order of the associated tuples, since

$$\underline{a} \geq \underline{b} \iff \forall_i \sum_{j=1}^{i} a_j \geq \sum_{j=1}^{i} b_j \iff \forall_i \ \widehat{a}_i \geq \widehat{b}_i.$$

The componentwise minimum of nondecreasing and concave tuples is nondecreasing and concave. Dominance therefore is a meet-semilattice order of the partitions and thus automatically a lattice order.

Example 1. $\underline{a} := 5+1+1+1$ and $\underline{b} := 4+2+2$ are partitions of 8. Their infimum can be computed via the associated tuples:

$$\begin{array}{rcl} \widehat{\underline{a}} = (0,5,6,7,8,\dots) & & \\ \widehat{\underline{b}} = (0,4,6,8,8,\dots) & & \\ \hline \min(\widehat{\underline{a}}, \widehat{\underline{b}}) = (0,4,6,7,8,\dots) & = & \widehat{\underline{c}} \ \text{ for } \ c = 4+2+1+1. \end{array}$$

Thus the infimum of $\underline{a}$ and $\underline{b}$ is

$$(5+1+1+1) \wedge (4+2+2) = 4+2+1+1.$$

Note that the componentwise maximum

$$\max(\widehat{\underline{a}}, \widehat{\underline{b}}) = (0,5,6,8,\dots)$$

of $\widehat{\underline{a}}$ and $\widehat{\underline{b}}$ is not concave, since $2 \cdot 6 \not\geq 5+8$, and does therefore *not* give the supremum.

The supremum of integer partitions can conveniently be computed using the notion of *duality*. The **dual** or **conjugate** partition $\underline{a}^*$ of a partition $\underline{a}$ is defined as

$$a_i^* := |\{j \mid a_j \geq i\}|, \quad i \in \{1, \dots, n\}.$$

A suggestive visualization is given by the **Ferrers diagrams**, which represent each partition of n by pebbles in an $n \times n$-matrix, one row for each summand. The dual partition corresponds to the transposed array, see Figure 2 for an example.

Dualization is order-reversing, as Brylawski has shown, and thus is a lattice anti-automorphism (of order 2, i.e., $x^{**} = x$ holds for all x). This allows to compute the supremum of integer partitions as

$$\underline{a} \vee \underline{b} = (\underline{a}^* \wedge \underline{b}^*)^*.$$

4+2+1 $\qquad$ $3+2+1+1=(4+2+1)^*$

Fig. 2. Ferrers diagrams for a partition of 7 and its dual.

Example 2. In the case of $\underline{a} := 5+1+1+1$ and $\underline{b} := 4+2+2$ we get

$$\begin{array}{rcl} \widehat{\underline{a}^*} &=& (0,4,5,6,7,8,\ldots) \\ \widehat{\underline{b}^*} &=& (0,3,6,8,8,8,\ldots) \\ \hline \min(\widehat{\underline{a}^*},\widehat{\underline{b}^*}) &=& (0,3,5,6,7,8,\ldots) \quad = \quad \widehat{\underline{c}^*} \text{ for } c = 5+2+1. \end{array}$$

The supremum of $\underline{a}$ and $\underline{b}$ therefore is

$$(5+1+1+1) \vee (4+2+2) = 5+2+1.$$

Ferrers diagrams can also visualize the dominance order relation. For that it is useful to rotate the diagrams counterclockwise by 90° and to think of them as *sand piles* (see e.g. [6]). "Rolling down" of some of the pebbles then leads to lower elements in the dominance order. Brylawski has stated this in a less pictorial but very helpful manner:

Proposition 1 ([3][1]). *A partition $\underline{b}$ is a lower neighbor of $\underline{a} := (a_i, \ldots, a_n)$ iff*

$$\underline{b} = (a_1, \ldots, a_i - 1, \ldots, a_j + 1, \ldots) \quad \textit{with } a_i - a_j = 2 \textit{ or } j = i+1.$$

3 The standard contexts

Brylawski used his characterization to determine the join-irreducible elements of the lattice. These are precisely the ones with a unique lower neighbor. The meet-irreducible partitions are the duals of these.

Proposition 2 ([3]). *The join-irreducible partitions of n are precisely the ones of the form*

$$(\underbrace{q+1,\ldots,q+1}_{r},\underbrace{q,\ldots,q}_{m-r},\underbrace{1,\ldots,1}_{a}),$$

where

$$n - a = q \cdot m + r, \quad a \geq 0, \quad m > r \geq 0, \quad q \geq 2, \quad \textit{and} \quad q \geq 3 \ \textit{ if } a \neq 0 \neq r.$$

[1] Greene and Kleitman [7] attribute the result to Muirhead [10]. Muirhead's paper indeed contains a related idea, but not in the form of our proposition.

Brylawski's result provides all the information necessary for writing down the standard contexts. Figure 3 shows the standard context of the partitions of 9. Actually, it also shows the standard contexts for $n = 1, \ldots, 8$, each of them as a square subcontext in the upper left corner. The sizes of these contexts can be obtained from Proposition 3.

Because of the lattice anti-automorphism of order 2 the standard contexts are symmetric and may be read as adjacency matrices of graphs, however, with some vertices having loops. An example is shown in Figure 4.

$\mathbb{K}_{\text{Part}(9)}$	1 1 1 1 1 1 1 1 1	8 1	2 1 1 1 1 1 1 1	6 3	2 2 2 2 1	7 1 1	3 1 1 1 1 1 1	7 2	3 3 3	4 1 1 1 1 1	6 1 1 1	3 2 2 2	5 4	5 2 2	4 4 1	4 2 2 1	5 1 1 1 1
9																	
2 1 1 1 1 1 1 1		×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×
8 1		×															
2 2 2 1 1 1		×		×	×	×		×	×	×	×	×	×	×	×	×	×
5 4		×		×				×					×				
3 1 1 1 1 1 1		×		×		×	×	×	×	×	×	×	×	×	×	×	×
7 1 1		×				×		×									
2 2 1 1 1 1 1		×		×	×	×	×	×	×	×	×	×	×	×	×	×	×
3 3 3		×		×		×		×	×				×	×	×		
6 1 1 1		×		×		×		×			×						
4 1 1 1 1 1		×		×		×		×		×	×		×	×	×	×	×
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3 3 1 1 1		×		×		×		×	×		×		×	×	×	×	×
3 2 2 2		×		×		×		×	×		×	×	×	×	×	×	
4 3 1 1		×		×		×		×			×		×	×	×		
5 1 1 1 1		×		×		×		×			×		×	×			×

Fig. 3. The standard context of the lattice of partitions of the integer 9. It contains the standard contexts for $n = 1, \ldots, 8$ as square subcontexts, each in the upper left corner. For the sizes of these subcontexts, cf. Proposition 3.

Proposition 3. *The number of join-irreducible integer partitions of n (as well as the number of meet-irreducible ones) is*

$$\left\lfloor \frac{(n+1) \cdot (n+2)}{6} \right\rfloor - 1.$$

The first values for $n = 1, \ldots, 15, \ldots$ are

$$0, 1, 2, 4, 6, 8, 11, 14, 17, 21, 25, 29, 34, 39, 44, \ldots.$$

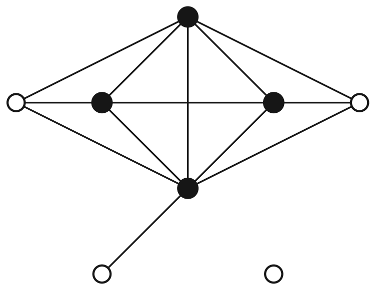

Fig. 4. The graph obtained from reading the standard context of the partitions of 6 as an adjacency matrix (the 8×8 matrix in the upper left of Fig. 3). The solid vertices have loops.

(Note that the n^{th} value is equal to

$$0 + 1 + 1 + 2 + 2 + 2 + 3 + 3 + 3 + 4 + 4 + 4 + 5 + \ldots,$$

where n is the number of summands.)

Proof. For given values of n, $a \geq 0$, and $m \geq 1$ there is at most one choice of q and r in Proposition 2, since

$$q = (n-a) \text{ div } m$$
$$r = (n-a) \bmod m.$$

The condition $q \geq 2$ requires that $m \leq \lfloor \frac{n-a}{2} \rfloor$. For $a \neq 0$ we must have $m \leq \lfloor \frac{n-a}{3} \rfloor$, except for the case that m divides $n-a$ (and thus $r = 0$). The latter happens only if $n-a$ is even and $m = \frac{n-a}{2}$. The number of solutions therefore is $\lfloor \frac{n}{2} \rfloor$ for $a = 0$, $\lfloor \frac{n-1}{2} \rfloor$ when $a \neq 0$, $m > \lfloor \frac{n-a}{3} \rfloor$ and $r = 0$, and $\sum_{a=1}^{n} \lfloor \frac{n-a}{3} \rfloor$ else. Because of $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n-1}{2} \rfloor = n-1$ and $\sum_{a=1}^{n} \lfloor \frac{n-a}{3} \rfloor = \sum_{a=0}^{n-1} \lfloor \frac{a}{3} \rfloor = \lfloor \frac{(n-1)(n-2)}{6} \rfloor$ we get for the number of solutions

$$n - 1 + \left\lfloor \frac{(n-1)(n-2)}{6} \right\rfloor,$$

as claimed in the proposition. □

4 Embeddings

A simple way to turn a partition $\underline{a} := (a_1, \ldots, a_n)$ of n into a partition of $n+1$ is to "add a 1 at the end[2]", i.e., to obtain $\underline{a} + 1 := (a_1, \ldots, a_{l(\underline{a})}, 1, 0, \ldots)$. The

[2] Recall that $l(\underline{a})$ denotes the highest index of a non-zero summand of $\underline{a}$

mapping

$$\underline{a} \mapsto \underline{a} + 1$$

obviously is 1-1. It satisfies

$$\underline{a} \geq \underline{b} \iff \underline{a} + 1 \geq \underline{b} + 1,$$

as can easily be seen by comparing the associated $(n+2)$-tuples

$$\widehat{\underline{a}+1} = (0, a_1, a_1 + a_2, \ldots, a_1 + \ldots + a_{l(\underline{a})}, n+1, \ldots)$$

$$\widehat{\underline{b}+1} = (0, b_1, b_1 + b_2, \ldots, b_1 + \ldots + b_{l(\underline{b})}, n+1, \ldots)$$

(recall that $\underline{a} \geq \underline{b}$ implies $l(\underline{a}) \leq l(\underline{b})$). When denoting the range of this mapping by

$$R_n := \{\underline{a} + 1 \mid \underline{a} \in \text{Part}(n)\} \subseteq \text{Part}(n+1),$$

we can conclude that the mapping $\underline{a} \mapsto \underline{a} + 1$ is an order isomorphism from $\text{Part}(n)$ to R_n, with the induced order.

Proposition 4. *The set R_n is infimum-closed, and so is its dual,*

$$R_n^* := \{\underline{a}^* \mid \underline{a} \in R_n\}.$$

Proof. Note that $\underline{a} \in R_n$ iff the associated $(n+2)$-tuple contains the entry n. Now let $\underline{a}$ and $\underline{b}$ be in R_n, and let $i(\underline{a}), i(\underline{b})$ be indices for which $\widehat{a}_{i(\underline{a})} = n = \widehat{b}_{i(\underline{b})}$. W.l.o.g. assume that $i(\underline{a}) \geq i(\underline{b})$. Then $\widehat{b}_{i(\underline{a})} \in \{n, n+1\}$, and the $i(\underline{a})^{\text{th}}$ entry of $\widehat{\underline{a} \wedge \underline{b}}$ is $\min(n, n+1) = n$. Thus $\underline{a} \wedge \underline{b} \in R_n$.

Similarly, the elements $\underline{a} \in R_n^*$ can be characterized as those satisfying $a_1 > a_2$, or, equivalently, $2 \cdot \widehat{a}_1 > \widehat{a}_2$. The first components of the associated $(n+2)$-tuple of $\underline{a}^* \wedge \underline{b}^*$ are 0, $\min(a_1, b_1)$, and $\min(a_1 + a_2, b_1 + b_2)$. And indeed it follows from $a_1 > a_2$ and $b_1 > b_2$ that $2 \cdot \min(a_1, b_1) > \min(a_1 + a_2, b_1 + b_2)$. Thus $\underline{a}^* \wedge \underline{b}^* \in R_n^*$. □

Lemma 1. *The mapping $\underline{a} \mapsto \underline{a} + 1$ is a lattice embedding of Part(n) into Part$(n+1)$, and so is its dual, $\underline{a} \mapsto (\underline{a}^* + 1)^*$.*

Proof. Proposition 4 implicitly states that R_n also is supremum-closed (and R_n^* as well). Because when $\underline{a}$ and $\underline{b}$ are in R_n, then $\underline{a}^*$ and $\underline{b}^*$ are in R_n^* and so is $\underline{a}^* \wedge \underline{b}^*$, according to the proposition. But then $\underline{a} \vee \underline{b} = (\underline{a}^* \wedge \underline{b}^*)^*$ is in R_n, as claimed. □

Note that R_n and R_n^* are sublattices, but *not* complete sublattices, since R_n does not contain the largest and R_n^* does not contain the smallest element of $\text{Part}(n+1)$.

The lemma tells us that the partition lattice of n contains that of $n-1$ in at least two copies. This raises the question if a recursive construction is possible, perhaps even for the standard contexts. We say that a **context embedding**

$$(H, N, J) \underset{\alpha}{\overset{\beta}{\rightrightarrows}} (G, M, I)$$

of (H, N, J) into (G, M, I) is a pair of one-to-one mappings $\alpha : H \to G$, $\beta : N \to M$ such that $h\ J\ n \iff \alpha(h)\ I\ \beta(n)$, and that a formal context (G, M, I) is **symmetric** if there are mappings $\delta_{GM} : G \to M$ and $\delta_{MG} : M \to G$, inverse to each other, such that

$$g\ I\ m \iff \delta_{MG}(m)\ I\ \delta_{GM}(g).$$

For simplification it is common to use the same symbol for both δ_{GM} and δ_{MG}, for example, the symbol "$*$". The symmetry condition then simplifies to

$$g\ I\ m \iff m^*\ I\ g^*.$$

Moreover, it is assumed that $x^{**} = x$ holds for all objects and all attributes x. Finally, when (H, N, J) and (G, M, I) are symmetric formal contexts, then a context embedding

$$(H, N, J) \overset{\beta}{\underset{\alpha}{\rightrightarrows}} (G, M, I)$$

is called a **symmetric context embedding** if

$$\beta(n) = \alpha(n^*)^*$$

holds for all $n \in N$. Graphically, this means that (G, M, I) can be written as a symmetric table (invariant under transposition) and that (H, N, J) is isomorphic to an induced subcontext of (G, M, I), which is itself symmetric under the same symmetry.

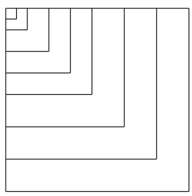

Fig. 5. The way the subcontexts of sizes 0,1,4,6,8,11,14,17 are nested in Figure 3.

The example for which we introduce these definitions is given in Figure 3. It shows the standard context of the lattice of partitions of the integer 9 as a symmetric table. It contains the standard contexts for $n = 1, \ldots, 8$ as symmetrically embedded subcontexts[3]. More precisely, it shows a nested series of symmetric embeddings, as indicated in Figure 5.

This recursive construction of the standard context for $n = 9$ shows some regularities. But it is not obvious, what the general construction rule might be. The reason for that is given by the next proposition.

[3] For $n = 1$, the standard context is empty. The object and attribute names apply only for $n = 9$.

Proposition 5. *There is no symmetric embedding of the standard context for the lattice of partitions of $n = 9$ into that for $n = 10$.*

The *proof* is by exhaustive search: we wrote a simple computer program that checked all possible cases, without finding a solution. □

5 Size and breadth

Take another look at Figure 1. Two properties of the lattice are eye-catching: it is planar, and has many irreducible elements. In fact, half of its elements are join-irreducible (21 of 42), and the number of meet-irreducibles is 21 as well (but note that there are 12 doubly irreducible and 12 doubly reducible elements).

These observations are misleading, since they do not reflect a general trend. It turns out that for larger n the lattices of integer partitions grow much faster than their standard contexts. And as a consequence they must contain large Boolean lattices as suborders, which in turn implies a high order dimension.

The **breadth** of a complete lattice is the number of atoms of the largest Boolean lattice that it contains as a suborder, and is (in the doubly founded case) at the same time the size of the largest contranominal scale that is an induced subcontext of its standard context. Recall that the k^{th} contranominal scale is

$$\mathbb{N}^c(k) := (\{1, \ldots, k\}, \{1, \ldots, k\}, \neq).$$

We will show that partition lattices have unbounded breadth. The bounds which we derive are however very weak.

We start with an example. Consider the case $n = 200$. It is known (we found the result in [12], where it is cited from [2]), that the number of partitions of 200 is

$$3\ 972\ 999\ 029\ 388.$$

From Proposition 3 we obtain that the number of join-irreducible partitions is

$$\left\lfloor \frac{201 \cdot 202}{6} \right\rfloor - 1 = 6766.$$

A result of Albano and Chornomaz [1] (see below) states that the concept lattice of a formal context (G, M, I) with $|G| = 6766$ can have at most

$$\frac{4}{6} \cdot 6766^3 = 206\ 492\ 708\ 730.66\ldots$$

elements, unless the context contains a contranominal scale $\mathbb{N}^c(4)$. Therefore the lattice of partitions of 200 must contain a 16-element Boolean lattice as a suborder, i.e., it has breadth at least 4.

This can be generalized. The strategy is to show that the growth of the number of partitions forces the standard contexts to contain contranominal scales of arbitrary size. The **partition function** $p(n)$ gives the number of partitions for each positive integer n. It is sequence a000041 in the OEIS [11], from where we cite the first values:

$$1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, \ldots$$

There is an explicit formula for $p(n)$, but it is too complicated to be useful here. For an introduction see Wilf's lecture notes [12], where also an asymptotic formula

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \qquad (n \to \infty)$$

can be found. For our purposes, an elementary lower bound suffices:

Theorem 1 (Maróti [9]).

$$p(n) \geq \frac{e^{2\sqrt{n}}}{14} \qquad \textit{for all } n.$$

The second ingredient is

Theorem 2 (Albano & Chornomaz [1]). *Let* $\mathbb{K} := (G, M, I)$ *be an* $\mathbb{N}^c(k)$*-free formal context with finite* G *and* $k \leq \frac{|G|}{2}$*. Then*

$$|\underline{\mathfrak{B}}(\mathbb{K})| \leq \frac{k}{(k-1)!} |G|^{k-1}.$$

Combining these two theorems yields our desired result that partition lattices have unbounded breadth:

Theorem 3. *For each integer* k *there is an integer* n_k *such that for each* $n \geq n_k$ *the lattice of partitions of* n *has breadth at least* k*.*

Note that a concept lattice $\underline{\mathfrak{B}}(\mathbb{K})$ has breadth $< k$ iff $\mathbb{K}$ is $\mathbb{N}^c(k)$-free.

Proof. Let n be an integer for which the partition lattice is $\mathbb{N}^c(k)$-free and $k \leq |G|/2$. Then the inequality of Theorem 2 must hold and must remain valid if we replace the left side by Maróti's lower bound and the size of G by $\frac{(n+1)^2}{6}$, which is an upper bound due to Proposition 3. This gives

$$\frac{e^{2\sqrt{n}}}{14} \leq \frac{k}{(k-1)!} \cdot \frac{(n+1)^{2k-2}}{6^{k-1}}.$$

This can be rewritten as

$$e^{2\sqrt{n}} \leq \frac{14k}{6^{k-1}(k-1)!} \cdot (n+1)^{2k-2},$$

further as

$$e^{\sqrt{n}} \leq \sqrt{\frac{14k}{6^{k-1}(k-1)!}} \cdot e^{(k-1)\log(n+1)},$$

and eventually leads to

$$\sqrt{n} \leq c_k + (k-1) \cdot \log(n+1),$$

where $c_k = \log\left(\sqrt{\frac{14k}{6^{k-1}(k-1)!}}\right)$. Since the square root grows faster than any multiple of the logarithm, the inequality can only be fulfilled by finitely many values of n, when k is fixed. □

As an example, let $k := 4$ and $n := 200$. Then $c_k = \log(\sqrt{\frac{14 \cdot 4}{6^3 \cdot 6}}) = -1.571$, $\sqrt{n} = 14.14$, and $(k-1) * \log(n+1) = 15.91$. The inequality holds, in contrast to the previous example, where we used exact values rather than approximations. But for $n = 210$, we get $\sqrt{n} = 14.491$ and $(k-1) * \log(n+1) = 16.056$, so that the inequality is violated.

6 Conclusion and outlook

The lattices of integer partitions can be viewed as concept lattices. We described their standard contexts by a simple rule, based on Brylawski's findings, and determined their sizes. It turns out that the lattices are of subexponential size wrt. their standard contexts, and thus are of unbounded breadth.

The next step in the mathematical study of these lattices could be the characterization of the arrow relations and, building on that, the classification of the possible subdirectly irreducible factors, see [5]. Figure 6 gives an impression, showing that the arrow relations at look very promising, at least for the case $n = 9$.

It remains open if the partition lattices can play a rôle in conceptual data analysis. This seems unplausible at first. But we found the many applications that integer partitions have within mathematics (cf. [3] or [12]) surprising as well.

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3 3 1 1 1		×		×		×		×	×	↗	×	⤢	×	×	×	×	×
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4 3 1 1		×		×		×		×	↗		×		×	×	×	⤢	↗
5 1 1 1 1		×		×		×		×			×		×	×	⤢	↙	×

Fig. 6. Same as in Figure 3, but with the arrow relations.

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