A Data Analysis Application of Formal Independence Analysis*

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Abstract. In this paper we present a new technique for the analysis of data tables by means of Formal Independence Analysis (FIA). This is an analogue of Formal Concept Analysis for the study of independence relations in data, instead of hierarchical relations. A FIA of a context produces, when possible, its block diagonalization by detecting pairs of sets of objects and attributes that are not mutually incident, or *tomoi*, that partition the context. In this paper we combine this technique with the exploration of contexts with entries in a semifield to find independent sets in contingency matrices. Specifically, we apply it to a number of confusion matrices issued from cognitive experiments to find evidences for the hypothesis of perceptual channels.

1 Introduction and Motivation

In this paper we derive a technique for data analysis from the recently introduced Formal Independence Analysis, (FIA) [11]. This is an analysis technique for formal contexts based on the description of certain pairs of subsets of objects and attributes called *tomoi*, e.g. *divisions*, which are unrelated through the incidence. We set out to demonstrate how these *tomoi* allow us to dissect the structure and information of certain matrices.

Independent Perceptual Channels. Miller and Nicely [4] posited that for certain human perceptual tasks—e.g. consonant perception—the underlying structure of confusion matrices provide evidence of the existence of *perceptual channels* associated with specific *perceptual features*. This work is aimed at providing a technique to make such channels evident with the goals and techniques of Lattice Theory.

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Specifically, consider the confusion matrix C_{ij} describing the results of an iterated classification experiment "when presented with stimulus i, and the (human) classifier answered response j." If the hypothesis of independent channels were true, we would expect this confusion matrix to be reordered by specific permutations of its rows and columns into a block diagonal form, more specifically, a *squared* block diagonal form. In this block-diagonal form, each block would describe the confusions *within* a perceptual channel, while confusions *outside* the channel would not be observed.

Reading Guide. In this paper we will use the recently developed FIA (Section 2.1) to obtain a block-diagonal form for confusion matrices, that leads to the independent virtual channel hypothesis of Miller and Nicely. This result actually stems from the consideration of a disjoint union of subcontexts decomposition technique already available from [2] that we relate to the notion of tomos and boolean tomoi lattice (Section 2.2). Our main results are the theoretical technique (Section 3.1) and the actual analyses carried out in the Miller and Nicely data (Section 3.2). We also provide a Discussion, a look into Further Work and some Conclusions.

2 Methods

2.1 Formal Independence Analysis

FIA was defined to complement the analysis of the information in formal contexts carried out by FCA, originally in terms of the hierarchical relation of formal concepts in terms of the inclusion between extents and intents. Instead, FIA targets the *relation of independence* between sets of objects and attributes [11], therefore called *tomoi*[3] The objects in the "extent" of a formal tomoi have no relation with the attributes of the "intent" of the tomoi.

Theorem 1 (Basic theorem of formal independence analysis).

1. *The context analysis phase: Given a formal context (G, M, I),*
   (a) *The operators $\cdot^{\sim} : 2^G \to 2^M$ and $\cdot_{\sim} : 2^M \to 2^G$*

$$\alpha^{\sim} = M \setminus \bigcup_{g \in \alpha} I(g, \cdot) = \{m \in M \mid g \not\!I m \text{ for all } g \in \alpha\} \tag{1}$$

$$\beta_{\sim} = G \setminus \bigcup_{m \in \beta} I(\cdot, m) = \{g \in G \mid g \not\!I m \text{ for all } m \in \beta\} \tag{2}$$

form a right-Galois connection $(\cdot^{\sim}, \cdot_{\sim}) : (2^G, \subseteq) \rightharpoonup (2^M, \subseteq)$ whose formal *tomoi are the pairs (α, β) such that $\alpha^{\sim} = \beta$ and $\alpha = \beta_{\sim}$.*

[3] From the Greek "tomos-tomoi", *division.*

(b) The set of formal tomoi $\mathfrak{A}(G,M,I)$ *with the relation*

$$(\alpha_1,\beta_1) \leq (\alpha_2,\beta_2) \quad \textit{iff} \quad \alpha_1 \supseteq \alpha_2 \quad \textit{iff} \quad \beta_1 \subseteq \beta_2$$

is a complete lattice, which is called the tomoi lattice of (G,M,I) *and denoted* $\underline{\mathfrak{A}}(G,M,I)$*, where infima and suprema are given by:*

$$\bigwedge_{t\in T}(\alpha_t,\beta_t) = \left(\bigcup_{t\in T}\alpha_t, \left(\bigcap_{t\in T}\beta_t\right)_{\sim}^{\sim}\right) \quad \bigvee_{t\in T}(\alpha_t,\beta_t) = \left(\left(\bigcap_{t\in T}\alpha_t\right)^{\sim}_{\sim}, \bigcup_{t\in T}\beta_t\right)$$

(c) The mappings $\overline{\gamma} : G \to \underline{\mathfrak{A}}(G,M,I)$ *and* $\overline{\mu} : M \to \underline{\mathfrak{A}}(G,M,I)$

$$g \mapsto \overline{\gamma}(g) = (\{g\}^{\sim}{}_{\sim}, \{g\}^{\sim}) \qquad m \mapsto \overline{\mu}(m) = (\{m\}_{\sim}, \{m\}_{\sim}{}^{\sim})$$

are such that $\overline{\gamma}(G)$ *is infimum-dense in* $\underline{\mathfrak{A}}(G,M,I)$ *,* $\overline{\mu}(M)$ *is supremum-dense in* $\underline{\mathfrak{A}}(G,M,I)$*.*

2. *The context synthesis phase: Given a complete lattice* $\mathbb{L} = \langle L, \leq\rangle$
   (a) $\mathbb{L}$ *is isomorphic to*[4] $\underline{\mathfrak{A}}(G,M,I)$ *if and only if there are mappings* $\overline{\gamma} : G \to L$ *and* $\overline{\mu} : M \to L$ *such that*
   – $\overline{\gamma}(G)$ *is infimum-dense in* $\mathbb{L}$ *,* $\overline{\mu}(M)$ *is supremum-dense in* $\mathbb{L}$*, and*
   – $g\ I\ m$ *is equivalent to* $\overline{\gamma}(g) \ngeq \overline{\mu}(m)$ *for all* $g \in G$ *and all* $m \in M$*.*
   (b) In particular, $\mathbb{L} \cong \underline{\mathfrak{A}}(L,L,\ngeq)$ *and, if* L *is finite,* $\mathbb{L} \cong \underline{\mathfrak{A}}(M(\mathbb{L}),J(\mathbb{L}),\ngeq)$ *where* $M(\mathbb{L})$ *and* $J(\mathbb{L})$ *are the sets of meet- and join-irreducibles, respectively, of* $\mathbb{L}$*.*

It is already known that the lattices of formal tomoi and concepts are deeply related [13,7]. Recall that the contrary context to any (G,M,I) is the context (M,G,I^{cd}), where the incidence has been transposed and inverted.

Proposition 1. *The formal lattice of the contrary formal context is isomorphic to the tomoi lattice:*

$$\underline{\mathfrak{A}}(G,M,I) \cong \underline{\mathfrak{B}}(M,G,I^{\mathrm{cd}})$$

2.2 Disjoint Context Sum and Adjoined Lattices

To set this scenario in a Formal Concept Analysis setting, recall from [2, Definition 30] that the *disjoint sum of two contexts* $\mathbb{K}_1 = (G_1,M_1,I_1)$ and $\mathbb{K}_2 = (G_2,M_2,I_2)$, with disjoint object and attribute sets is the context $\mathbb{K}_1 \dot{\cup} \mathbb{K}_2 = (G_1 \cup G_2, M_1 \cup M_2, I_1 \cup I_2)$, and that the concept lattice of the total context is the *horizontal sum of the two concept lattices*, that is, a union of the two lattices which only overlap in the top and bottom elements

$$\mathbb{K} = \mathbb{K}_1 \dot{\cup} \mathbb{K}_2 \iff \underline{\mathfrak{B}}(\mathbb{K}_1 \dot{\cup} \mathbb{K}_2) = \underline{\mathfrak{B}}(\mathbb{K}_1) \dot{\cup} \underline{\mathfrak{B}}(\mathbb{K}_2).$$

[4] Read *can be built as.*

This can be straightforwardly generalized to a finite number n of lattices,

$$\mathbb{K} = \dot{\bigcup}_{i=1}^{n} \mathbb{K}_i \iff \underline{\mathfrak{B}}\left(\dot{\bigcup}_{i=1}^{n} \mathbb{K}_i\right) = \dot{\bigcup}_{i=1}^{n} \underline{\mathfrak{B}}(\mathbb{K}_i). \tag{3}$$

This is what we call in this paper an *(explicit) block diagonal form* for the context, which results in a concept lattice of *adjoined sublattices*.

For this latter generalization, notice that each extent of $\mathbb{K}$, except for the extent $G = \cup_i G_i$, is entirely contained in one of the sets G_i, and concept-lattice dually for intents. So it makes sense to say that *two non-extreme concepts are orthogonal is they belong to different adjoined sublattices*[5].

The relationship between tomoi lattices and block decompositions is provided by the following proposition.

Proposition 2. *If* $\underline{\mathfrak{A}}(G, M, I) \cong 2^n$ *then the context* (G, M, I) *has an explicit block diagonal form.*

Proof (Sketch). By Theorem 1 (item 2.b) the context (G, M, I) can be transformed into another one whose object-concepts are the meet-irreducible elements and whose attribute-concepts are the join-irreducible elements and, hence, because of the isomorphism with 2^n, they are the co-atoms and the atoms, respectively. Moreover, they are complementary pairs of one object-tomos and one attribute-tomos.

As consequence, it is possible to reorganize the tabular expression of (G, M, I) in such a way that we obtain a block diagonal form. □

3 Results

3.1 Theoretical Analysis.

The purpose of proving the existence of independent channels for different percepts can be achieved by reducing a confusion matrix to a block diagonal form. But, confusion matrices are *not* binary incidences and may not be subject to a simple process of block diagonalization. Instead, we may look for an *approximate block-diagonal block*, that retains the main structure of the confusions.

We can motivate this approximation in the following way:

- A perfect classifier would obtain a diagonal matrix of counts. This has been proven in terms of information-theoretic arguments in [9], for instance.
- But in most cases what we can hope for is a diagonally dominant matrix, that is not even symmetrical. For instance, the heatmap of the symmetrized confusion matrix for the M&N data for −6dB, to the left of Fig. 1, shows such a shape. Even its symmetrical part of C_S has a corresponding structure that is far from being block-diagonal, e.g. center of Fig. 1.

[5] The basis for this definition is, of course, the embedding of extents and intents as vectors in semimodules over an idempotent semifield which allows us to define a dot product between extents, resp. intents. [10]. Note that in idempotent semimodules, which are *zero-sum free*, null dot-products can only occur for vectors of disjoint support, and this is precisely the case at hand.

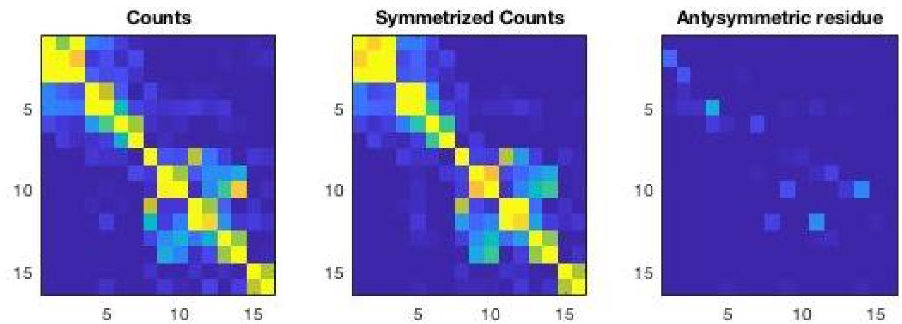


Fig. 1: (Color online) Heatmaps of the count confusion matrix in M&N for a SNR of -6dB. Left: count matrix; C center: symmetrized count matrix C_S; right: antisymmetrical residue C_A.

- Using *structural analysis* from an adequately transformed matrix $M = f(C_A)$ we could use the paradigm of *Landscapes-of-Knowledge (LoK)* [14] extended to multi-valued contexts [8] to explore the sequence of boolean incidences $I(\varphi)_{ij} = M_{ij} \lesseqgtr \varphi$ where φ ranges in the values of the original matrix:
  - Choosing $I(\varphi)_{ij} = M_{ij} \geq \varphi$ uses the *min-plus structural analysis*, while
  - Choosing $I(\varphi)_{ij} = M_{ij} \leq \varphi$ uses the *max-plus structural analysis*.
- The criterion for finding a “correct” value for φ is to ensure that the $I(\varphi)$ has a tomoi lattice that is boolean. A proxy criterion for this is to select and inspect only those φ whose number of formal tomoi is a power of 2. Note that after obtaining the appropriate φ by Proposition 2 we would have the block-decomposition.

In the following section, we check the feasibility of this scheme on the Miller and Nicely data.

3.2 FIA Exploration of Confusion Matrices

Data Description. In this paper we will use the data from the Miller and Nicely study to show examples of phenomena and test the proposed data analysis procedures. These are the confusion data of a consonant perception task, and we will refer to it as the *M&N data*. Specifically they are six different confusion matrices of 16 entries for different Signal-to-Noise Ratios (SNR) in dB of $\{-18, -12, -6, 0, 6, 12\}$ obtained in a (human) speech recognition task for the consonants listed in Table 1. The stimuli where balanced, but the responses may be unbalanced due to non-symmetrical confusion effects.

Data Preprocessing. Due to the symmetry inherent in the confusion task, since the category of the responses was the same as that of the stimuli, we extracted the symmetric component of each confusion matrix. This was done by obtaining from each matrix C its symmetric component $C_S = (C + C^{\mathrm{t}})/2$.

Table 1: Ordering of the consonants used in the confusion matrices analyzed (from [4]).

symbol	p , t , k , f , th , s , sh , b , d , g , v , dh , z , zh , m , n
phone	/p/, /t/, /k/, /f/, /θ /, /s/, /$\int$/, /b/, /d/, /g/, /v/, /ð/, /z/, /zh/, /m/, /n/

The antisymmetric component $C_A = (C - C^{\mathrm{t}})/2$ can then be interpreted as a residue. For the M&N confusion matrix at -6dB these two components can be seen in Fig. 1.

The data were preprocessed to obtain both the Pointwise Mutual Information (MI) and the Weighted Pointwise Mutual information (WPMI) as shown in Fig. 2. Although prior work suggested that WPMI lends itself to more clear

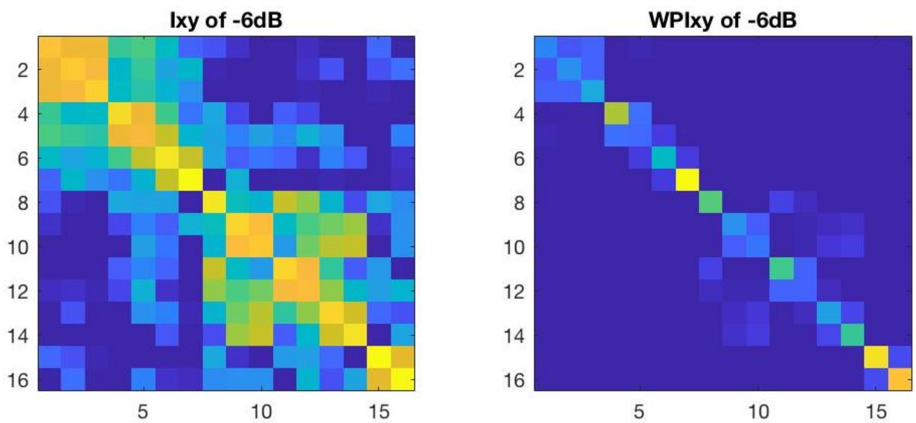


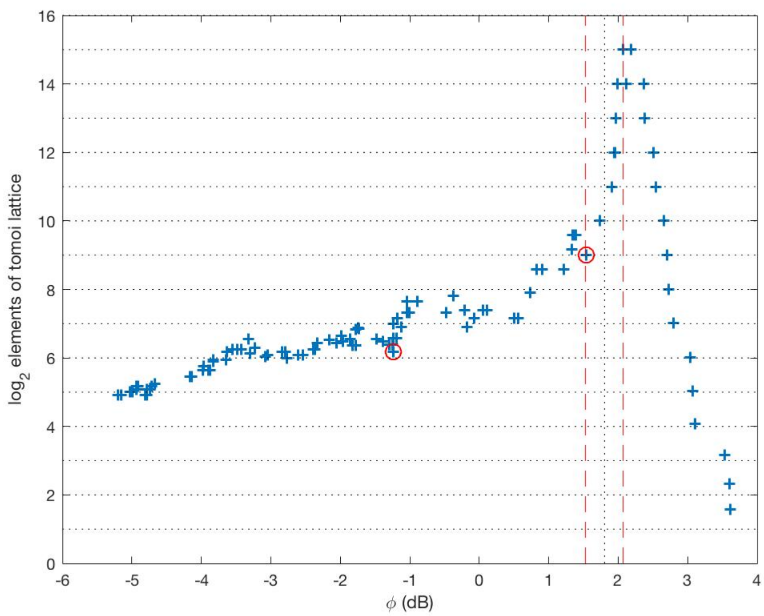
Fig. 2: (Color online) Heatmaps of the confusion matrix in M&N for a SNR of -6dB for different preprocessing. Left: pointwise mutual information. Right: weighted pointwise mutual information.

analyses, for the purpose of finding independent blocks in the matrix, we found—on using both types of data preprocessing—MI to retain more details about confusions that define the blocks, e.g. between elements that share (unknown) features motivating the confusion, for instance the voiceless fricatives /s/ vs. /$\int$/.

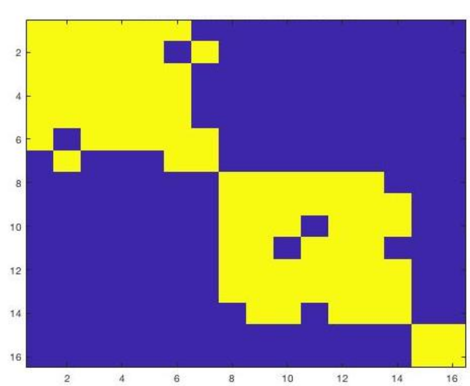
Data Analysis. We carried out min-plus exploratory analysis in the MI-transformed confusion matrix above by thresholding for each φ in increasing order and generated a sequence of $K = 105$ (binary) formal contexts $\mathbb{K}(\varphi_k) = (G, M, I(\varphi_k)), k \in [1, \ldots K]$.

For each of these contexts, we calculated the number of formal tomoi for each thresholded $I(\varphi)$ by actually working out the formal concepts of the contrary context $\mathbb{K}(\varphi)^{\mathrm{cd}} = (M, G, \not{I}^{\mathrm{t}})$. We do not explore at $\varphi = -\infty$ which entails a trivial full-incidence and a count of one tomoi.

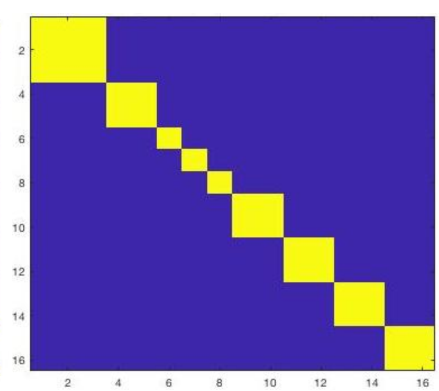
The graph of these counts in base-2 logarithm, shown in Fig. 3.a, allows us to define three regions:



(a) Tomoi count of $I(\varphi)$ vs. φ(dB)



(b) $I(\varphi)$ at $\varphi = -1.299010$



(c) $I(\varphi)$ at $\varphi = 1.539160$

Fig. 3: (Color online) Number of formal tomoi vs φ for $I(\varphi)$ and heatmaps for two highlighted φ.

– *An **initial segment** where the threshold is too lax and we see essentially few blocks and a number of "noise" tomoi, where our assumption, viz. that there are virtual channels, does not hold.*
  In the example being analyzed, this is the range $(-5.2, 1.53)$, to the left of the leftmost vertical line in Fig. 3.a . To ascertain the shapes of the thresholded we present an instance for where $\varphi \approx -1.23$ and $|\mathbb{K}(\varphi)| = 30$ focused on by the leftmost circle. In the heatmap of Fig. 3.b we can see and inkling of three different blocks, but since they are not complete, a number of "noisy" tomoi appear, making the tomoi lattice drift away from 2^3. Figure 4.a shows this non-boolean tomoi lattice whose incidence is that of Fig. 3.b
– *A **middle segment** where we start seeing many blocks, and consequently the number of tomoi $|\underline{\mathfrak{A}}(G, M, I(\varphi))|$ falls exactly into one of the powers of 2, where our assumption holds.*
  In the example, this is $\varphi \in [1.53, 2.07]$ between the vertical lines in Fig. 3.a comprising the ramp where the cardinalities range from 2^9 to 2^{14} tomoi. This is the case, for instance, of $\varphi \approx -1.53, |\mathbb{K}(\varphi)| = 2^9$, signaled as the rightmost red circle. We can see the 9-block incidence in Fig. 3.c, while Fig. 4.b shows the boolean lattice $\mathbb{K}(\varphi) \cong 2^9$. For reference, the (average) mutual information for this matrix, $MI_{-6dB} = 1.80$ falls within this range, and would generate the tomoi lattice isomorphic to 2^{10}.
– *A **final segment** where the threshold is too stringent and we no longer see a block diagonal form.*
  This is the least interesting zone for us. In the example it appears as a descending slope in the range $\varphi \in (2.07, 3.61)$ of Fig. 3.a.

We checked whether this behavior was analogous for all confusion matrices by analyzing the rest of the matrices at different SNR. The following are the main trends of analysis:

– We could only obtain boolean tomoi lattices considering all stimuli for those confusion matrices with SNR of $\{18, 12, 6, 0, -6\}$. The matrices at SNR $\in \{-12, -18\}$ were too noisy and some elements in the diagonal were less stable than elements off the diagonal, hence they disappeared on early exploration.
– In all of the instances where in some range of MI values the exploration procedure obtained boolean tomoi lattices, the average MI for the whole matrix, that is in the standard definition of mutual information, actually belonged in the range where the hypothesis held. Most of the times, this MI was close to the value for values of φ that obtained the boolean tomoi lattice of highest cardinality.
– The highest SNR in the confusion matrix being analyzed, the higher number of blocks in $I(\varphi)$. This is congruent with the supposition that high SNR situations allow us to distinguish individual phones better and it is therefore more difficult to obtain evidence of the perceptual channels through confusions.

Extracting Perceptual Channels. The tomoi provide the basis for obtaining the perceptual channels on top of boolean tomoi lattices, since for every object-tomoi, a *meet-irreducible*, its complement is an attribute tomoi, hence a *join-irreducible*. By the properties of complementary tomoi, the crossed extents and intents, define the blocks in the block diagonalization.

To see this, consider Table 2 of object-tomoi *extents* and their complementary tomoi intents—the attribute tomoi—to be used to build the block-diagonal form of (3). We see how, modulo a permutation, they constitute a refinement of the perceptual channels that Miller and Nicely proposed [4].

Table 2: Paired table of meet- and join-irreducibles of $\mathbb{K}(\varphi)$ in Fig. 4.b .

Object and attribute subsets	object-tomoi extent	complementary attribute-tomoi intent
(G_1, M_1)	$\{p, t, k\}$	$\{p, t, k\}$
(G_2, M_2)	$\{f, \theta\}$	$\{f, \theta\}$
(G_3, M_3)	$\{s\}$	$\{s\}$
(G_4, M_4)	$\{\int\}$	$\{\int\}$
(G_5, M_5)	$\{b\}$	$\{b\}$
(G_6, M_6)	$\{d, g\}$	$\{d, g\}$
(G_7, M_7)	$\{v, ð\}$	$\{v, ð\}$
(G_8, M_8)	$\{z, zh\}$	$\{z, zh\}$
(G_9, M_9)	$\{m, n\}$	$\{m, n\}$

Discussion and Further Work. Note that the problem we address in this paper was already approached in [12], but not solved satisfactorily, and we believe FIA provides a principled approach to the study of independent blocks within matrices.

In fact, FIA seems to detect much finer perceptual channels than the original paper suggests, perhaps because of the granularity of the perceptual features used there (see below). In order to obtain a rougher partition of phones to support Miller and Nicely's hypothesis, we have tried to analyze a balanced mixture of all the confusion matrices. But FIA has proven too strong for this unrealistic type of noise: the absence of confusions at 12dB dominates the behavior of the mixture, and the confusions from those behaviors at -18dB and -12dB are lost. Recall that it is precisely from the confusions where we obtain the evidence for the perceptual channels, so clearly a more nuanced approach to such mixture would be needed.

Although we have provided a data-induced procedure to obtain perceptual channels from confusion matrices, this is only a first step in actually obtaining the *experimental channels*. In particular, we have not investigated justifying those channels in terms of perceptual characteristics. While Miller and Nicely proposed an encoding of phones based on traditional *categorical* phonetic features, modern studies favor the consideration of *numeric* features. In our opinion this necessarily entails considering idempotent semimodule models of such spaces [6] and would lead to higher-value hypotheses. This is left for future work.

By no means is ours the only attempt at block-diagonalizing matrices over idempotent semifields. In fact, such process is important for the calculation of the Moore-Penrose inverse of a matrix over an idempotent semifield [5]. The main difference with out work is that we are trying to *approximate* the block-diagonal form in the presence of an *implicit noise*.

Yet a more general version of the problem is that of Cell Formation (CF) in Group Technology, in the field of Manufacturing [1], because it involves the block diagonalization of rectangular matrices. FIA is not restricted to squared matrices, but our application and interpretation indeed are because of the fact that confusion matrices are usually square. CF therefore opens up as an open research and application avenue for FIA.

Finally, further work is necessary to ascertain the relationship of lattices of formal tomoi to lattices of formal concepts, as well as to find out whether these are the only information lenses available for formal contexts, or how to measure the "quality" of the tomoi, in an effort similar to that shown for triadic analysis and triclustering in [3]. Our next main aim, though, is to incorporate these techniques in the over-arching exploratory data analysis framework first laid out in full in [8].

4 Conclusions

We have introduced a new technique to analyze data tables based on the newly proposed Formal Independence Analysis. The purpose of the technique is to obtain *tomoi*,—pairs of sets of objects and attributes unrelated through an incident relation— and their complements in the lattice of tomoi, which define as many partitions of the sets of objects and attributes. These tomoi will then be used to define a block-diagonal form for the incidence.

We apply the technique to the diagonally-dominant incidences of confusion matrices. By a process of exploration we select special thresholds that obtain boolean tomoi lattices. In these lattices we obtain the meet-irreducible object-tomoi and their complements, the join-irreducible attribute-tomoi, that define diagonal blocks on the looked-for incidence.

These diagonal blocks can be interpreted as virtual channels that transmit different types of information in the spirit of some classical perceptual experiments, e.g. Miller and Nicely's.

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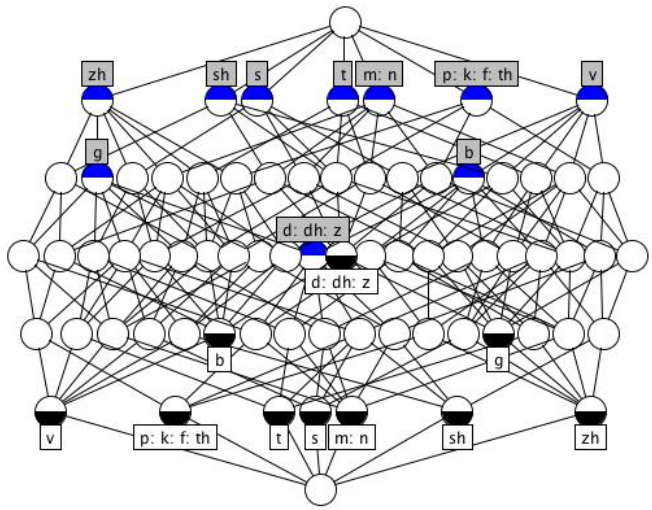
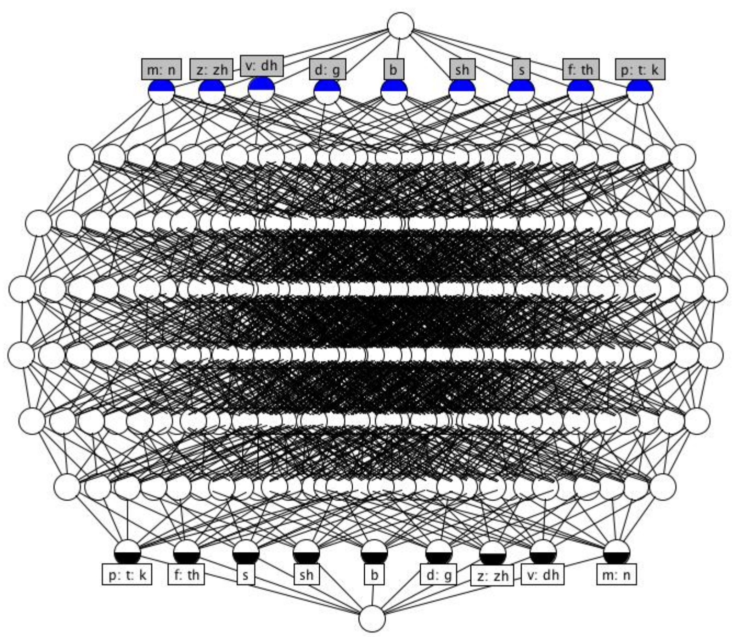


(a) Tomoi lattice for $\varphi = -1.299010$



(b) Tomoi lattice for $\varphi = 1.539160$

Fig. 4: (Color online) Tomoi lattices for the two incidences of Fig. 3