On the Existence of Right Adjoints for Surjective Mappings between Fuzzy Structures

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Abstract. We continue our study of the characterization of existence of adjunctions (isotone Galois connections) whose codomain is insufficiently structured. This paper focuses on the fuzzy case in which we have a fuzzy ordering ρ_A on A and a surjective mapping $f\colon \langle A, \approx_A\rangle \to \langle B, \approx_B\rangle$ compatible with respect to the fuzzy equivalences $\approx_A$ and $\approx_B$. Specifically, the problem is to find a fuzzy ordering ρ_B and a compatible mapping $g\colon \langle B, \approx_B\rangle \to \langle A, \approx_A\rangle$ such that the pair (f, g) is a fuzzy adjunction.

1 Introduction

Adjunctions, also called isotone Galois connections, are often used in mathematics in order to relate two (apparently disparate) theories, allowing for mutual cooperative advantages.

A number of papers are being published on the applications (both theoretical and practical) of Galois connections and adjunctions. One can find mainly theoretical papers [10, 15, 17, 23], as well as general applications to computer science, some of them dated more than thirty years ago [21] and, obviously, some more recent works on specific applications, such as programming [16, 22], data analysis [26], or logic [18, 25].

The study of new properties of Galois connections found an important niche in the theory of Formal Concept Analysis (FCA) and its generalizations, since the derivation operators which are used to define the formal concepts actually are a Galois connection. Just to name a few, Lumpe and Schmidt [20] consider adjunctions and their concept posets in order to define a convenient notion of morphism between pattern structures; Bělohlávek and Konečný [3] stress on the "duality" between isotone and antitone Galois connections in showing a case of mutual reducibility of the concept lattices generated by using each type of connection; Denniston *et al* [8] show how new results on Galois connection are applied to formal concept analysis, etc.

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It is certainly important to detect when an adjunction (or Galois connection) exists between two structured sets, and this problem has been already studied in the abstract setting of category theory. A different problem arises when either the domain or the codomain is unstructured: the authors studied in a previous work [14] the existence and construction of the right adjoint to a given mapping f in the general framework in which a mapping $f\colon A \to B$ from a (pre-)ordered set A into an unstructured set B is considered, aiming at characterizing those situations in which B can be (pre-)ordered and an isotone mapping $g\colon B \to A$ can be built such that the pair (f, g) is an adjunction. The general approach to this problem adopted in [14] was to consider the canonical decomposition of f with respect to the kernel relation, and consider the three resulting cases separately: the projection on the quotient, the isomorphism between the quotient and the image, and the final inclusion of the image into the codomain. The really important parts of the proof were the first and the last ones, since the intermediate part is straightforward.

We consider this work as an extension of the previous problem to a fuzzy framework, in which several papers on fuzzy Galois connections or fuzzy adjunctions have been written since its introduction by Bělohlávek in [1]; consider for instance [4, 9, 19, 27] for some recent generalizations. Some authors have introduced alternative approaches guided by the intended applications: for instance, Shi *et al* [24] introduced a definition of fuzzy adjunction for its use in fuzzy mathematical morphology.

In this paper, on the one hand, we will consider mappings compatible with fuzzy equivalences $\approx_A$ and $\approx_B$ defined on A and B respectively and, on the other hand, we will just focus on the first part of the canonical decomposition. This means that, up to isomorphism, we have a fuzzy ordering ρ_A on A and a *surjective* mapping $f\colon \langle A, \approx_A\rangle \to \langle B, \approx_B\rangle$ compatible with respect to the fuzzy equivalences $\approx_A$ and $\approx_B$. Specifically, the problem is to characterize when there exists a fuzzy ordering ρ_B and a compatible mapping $g\colon \langle B, \approx_B\rangle \to \langle A, \approx_A\rangle$ such that the pair (f, g) is a fuzzy adjunction.

2 Preliminaries

The most usual underlying structure for considering fuzzy extensions of Galois connections is that of complete residuated lattice, $\mathbb{L} = (L, \leq, \top, \bot, \otimes, \to)$. As usual, supremum and infimum will be denoted by $\vee$ and $\wedge$ respectively. An $\mathbb{L}$-fuzzy set in the universe U is a mapping $X\colon U \to L$ where $X(u)$ means the degree in which u belongs to X. Given X and Y two $\mathbb{L}$-fuzzy sets, X is said to *be included in* Y, denoted as $X \subseteq Y$, if $X(u) \leq Y(u)$ for all $u \in U$.

An $\mathbb{L}$-*fuzzy binary relation* on U is an $\mathbb{L}$-fuzzy subset of $U \times U$, that is $R\colon U \times U \to L$, and it is said to be:

- *Reflexive* if $R(a, a) = \top$ for all $a \in U$.
- $\otimes$-*Transitive* if $R(a, b) \otimes R(b, c) \leq R(a, c)$ for all $a, b, c \in U$.
- *Symmetric* if $R(a, b) = R(b, a)$ for all $a, b \in U$.

From now on, when no confusion arises, we will omit the prefix "$\mathbb{L}$-".

Definition 1. *A **fuzzy preordered set** is a pair $\mathbb{A} = \langle A, \rho_A \rangle$ in which ρ_A is a reflexive and $\otimes$-transitive fuzzy relation on A.*

Definition 2. *Let $\mathbb{A} = \langle A, \rho_A \rangle$ be a fuzzy preordered set. The extensions to the fuzzy setting of the notions of **upset** and **downset** of an element $a \in A$ are defined by $a^{\uparrow}, a^{\downarrow} \colon A \to L$ where*

$$a^{\downarrow}(u) = \rho_A(u, a) \quad \textit{and} \quad a^{\uparrow}(u) = \rho_A(a, u) \quad \textit{for all}\, u \in A.$$

Definition 3. *An element $m \in A$ is a **maximum** for a fuzzy set $X \colon A \to L$ if*

1. *$X(m) = \top$ and*
2. *$X \subseteq m^{\downarrow}$, i.e., $X(u) \leq \rho_A(u, m)$ for all $u \in A$.*

The definition of minimum is similar.

Since the maximum (respectively, minimum) of a fuzzy set needs not be unique, we will include special terminology for them: the crisp set of maxima, respectively minima, for X will be denoted p-max(X), respectively p-min(X).

Definition 4. *Let $\mathbb{A} = \langle A, \rho_A \rangle$ and $\mathbb{B} = \langle B, \rho_B \rangle$ be fuzzy preordered sets.*

1. *A mapping $f \colon A \to B$ is said to be **isotone** if $\rho_A(a_1, a_2) \leq \rho_B(f(a_1), f(a_2))$ for all $a_1, a_2 \in A$.*
2. *A mapping $f \colon A \to A$ is said to be **inflationary** if $\rho_A(a, f(a)) = \top$ for all $a \in A$.*
   *Similarly, f is **deflationary** if $\rho_A(f(a), a) = \top$ for all $a \in A$.*

From now on, we will use the following notation: For a mapping $f \colon A \to B$ and a fuzzy subset Y of B, the fuzzy set $f^{-1}(Y)$ is defined as $f^{-1}(Y)(a) = Y(f(a))$, for all $a \in A$.

The definition of fuzzy adjunction given in [11] was the expected extension of that in the crisp case. Namely,

Definition 5. *Let $\mathbb{A} = \langle A, \rho_A \rangle$, $\mathbb{B} = \langle B, \rho_B \rangle$ be fuzzy orders, and two mappings $f \colon A \to B$ and $g \colon B \to A$. The pair (f, g) forms a **fuzzy adjunction** between A and B, denoted $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$ if, for all $a \in A$ and $b \in B$, the equality $\rho_A(a, g(b)) = \rho_B(f(a), b)$ holds.*

As in the crisp case, there exist alternative definitions which are summarized in the theorem below:

Theorem 1 (See [11]). *Let $\mathbb{A} = \langle A, \rho_A \rangle$, $\mathbb{B} = \langle B, \rho_B \rangle$ be two fuzzy preordered sets, respectively, and $f \colon A \to B$ and $g \colon B \to A$ be two mappings. The following statements are equivalent:*

1. *$(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$.*
2. *f and g are isotone, $g \circ f$ is inflationary, and $f \circ g$ is deflationary.*
3. *$f(a)^{\uparrow} = g^{-1}(a^{\uparrow})$ for all $a \in A$.*

4. $g(b)^{\downarrow} = f^{-1}(b^{\downarrow})$ *for all* $b \in B$.
5. f *is isotone and* $g(b) \in \text{p-max}\, f^{-1}(b^{\downarrow})$ *for all* $b \in B$.
6. g *is isotone and* $f(a) \in \text{p-min}\, g^{-1}(a^{\uparrow})$ *for all* $a \in A$.

In the rest of this section, we introduce the preliminary definitions and results needed to establish the new structure we will be working on.

Definition 6. *A fuzzy relation* $\approx$ *on* A *is said to be a:*

– ***Fuzzy equivalence relation*** *if* $\approx$ *is a reflexive,* $\otimes$*-transitive and symmetric fuzzy relation on* A.
– ***Fuzzy equality*** *if* $\approx$ *is a fuzzy equivalence relation satisfying that* $\approx(a, b) = \top$ *implies* $a = b$*, for all* $a, b \in A$.

We will use the infix notation for a fuzzy equivalence relation, that is, we will write $a_1 \approx a_2$ instead of $\approx(a_1, a_2)$.

Definition 7. *Given a fuzzy equivalence relation* $\approx\colon A \times A \to L$*, the* ***equivalence class*** *of an element* $a \in A$ *is the fuzzy set* $[a]_{\approx}\colon A \to L$ *defined by* $[a]_{\approx}(u) = (a \approx u)$ *for all* $u \in A$.

Remark 1. Note that $[x]_{\approx} = [y]_{\approx}$ if and only if $(x \approx y) = \top$: on the one hand, if $[x]_{\approx} = [y]_{\approx}$, then $(x \approx y) = [x]_{\approx}(y) = [y]_{\approx}(y) = \top$, by reflexivity; conversely, if $(x \approx y) = \top$, then $[x]_{\approx}(u) = (x \approx u) = (y \approx x) \otimes (x \approx u) \leq (y \approx u) = [y]_{\approx}(u)$, for all $u \in A$; the other inequality follows similarly.

Definition 8 (See [6]). *Given a fuzzy equivalence relation* $\approx_A$ *on* A*, a fuzzy binary relation* $\rho_A\colon A \times A \to L$ *is said to be*

– $\approx_A$***-reflexive*** *if* $(a_1 \approx_A a_2) \leq \rho_A(a_1, a_2)$,
– $\otimes$-$\approx_A$***-antisymmetric*** *if* $\rho_A(a_1, a_2) \otimes \rho_A(a_2, a_1) \leq (a_1 \approx_A a_2)$,

for all $a_1, a_2 \in A$.

Definition 9. *A triplet* $\mathcal{A} = \langle A, \approx_A, \rho_A \rangle$ *in which* $\approx_A$ *is a fuzzy equivalence relation and* ρ_A *is* $\approx_A$*-reflexive,* $\otimes$*-*$\approx_A$*-antisymmetric and* $\otimes$*-transitive will be called* $\otimes$-$\approx_A$- fuzzy preordered set *or* ***fuzzy preorder with respect to*** $\approx_A$.

Observe that a fuzzy preorder relation wrt $\approx_A$ is a fuzzy preorder relation because $\top = (a \approx_A a) \leq \rho_A(a, a)$, therefore $\rho_A(a, a) = \top$, for all $a \in A$.

Definition 10. *Let* $\approx_A$ *and* $\approx_B$ *be fuzzy equivalence relations on the sets* A *and* B*, respectively. A mapping* $f\colon A \to B$ *is said to be* ***compatible*** *with* $\approx_A$ *and* $\approx_B$ *if* $(a_1 \approx_A a_2) \leq (f(a_1) \approx_B f(a_2))$ *for all* $a_1, a_2 \in A$.

3 On fuzzy adjunctions wrt fuzzy equivalences

The main idea to extend the notion of fuzzy adjunction to take into account fuzzy equivalences, namely, a fuzzy adjunction between $\mathcal{A} = \langle A, \approx_A, \rho_A \rangle$ and $\mathcal{B} = \langle B, \approx_B, \rho_B \rangle$ is, of course, to require f and g to be compatible mappings and include the necessary adjustments due to the use of fuzzy equivalences. A reasonable possibility is the following:

Definition 11. *Let $\mathcal{A} = \langle A, \approx_A, \rho_A \rangle$ and $\mathcal{B} = \langle B, \approx_B, \rho_B \rangle$ be two fuzzy preordered sets wrt $\approx_A$ and $\approx_B$, respectively. Let $f\colon A \to B$ and $g\colon B \to A$ be two mappings which are compatible with $\approx_A$ and $\approx_B$. The pair (f, g) is said to be a* ***fuzzy adjunction*** *between $\mathcal{A}$ and $\mathcal{B}$ if the following conditions hold*

(A1) $(a_1 \approx_A a_2) \otimes \rho_A(a_2, g(b)) \leq \rho_B(f(a_1), b)$
(A2) $(b_1 \approx_B b_2) \otimes \rho_B(f(a), b_1) \leq \rho_A(a, g(b_2))$

for all $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$.

Surprisingly, it turns out that Definitions 5 and 11 are very closely related, in fact, they are equivalent up to compatibility of the mappings.

Theorem 2. *Let $\mathcal{A} = \langle A, \approx_A, \rho_A \rangle$ and $\mathcal{B} = \langle B, \approx_B, \rho_B \rangle$ be two fuzzy preordered sets wrt $\approx_A$ and $\approx_B$, respectively. Let $f\colon A \to B$ and $g\colon B \to A$ be two mappings which are compatible with $\approx_A$ and $\approx_B$, respectively.*

Then, the pair (f, g) is a fuzzy adjunction between $\mathcal{A}$ and $\mathcal{B}$ if and only if $\rho_A(a, g(b)) = \rho_B(f(a), b)$ for all $a \in A$ and $b \in B$.

Proof. Assume that for all $a \in A$ and $b \in B$ the equality $\rho_A(a, g(b)) = \rho_B(f(a), b)$ holds.

Let $a_1, a_2 \in A$ and $b \in B$. Since f is a map which is compatible with $\approx_A$ and $\approx_B$, then

$$(a_1 \approx_A a_2) \otimes \rho_A(a_2, g(b)) \leq (f(a_1) \approx_B f(a_2)) \otimes \rho_A(a_2, g(b)).$$

By the hypothesis, we obtain that

$$(f(a_1) \approx_B f(a_2)) \otimes \rho_A(a_2, g(b)) \leq (f(a_1) \approx_B f(a_2)) \otimes \rho_B(f(a_2), b).$$

As ρ_B is $\approx_B$-reflexive and transitive, we have that

$$(f(a_1) \approx_B f(a_2)) \otimes \rho_B(f(a_2), b) \leq \rho_B(f(a_1), f(a_2)) \otimes \rho_B(f(a_2), b) \leq \rho_B(f(a_1), b).$$

Therefore, $(a_1 \approx_A a_2) \otimes \rho_A(a_2, g(b)) \leq \rho_B(f(a_1), b)$ for all $a_1, a_2 \in A$ and $b \in B$. Analogously, the condition $(A2)$ holds.

Conversely, assume now that conditions $(A1)$ and $(A2)$ hold. Applying condition $(A1)$, for $a \in A$ and $b \in B$, we have that $(a \approx_A a) \otimes \rho_A(a, g(b)) \leq \rho_B(f(a), b)$. Being $\approx_A$ reflexive, it is deduced that $\rho_A(a, g(b)) \leq \rho_B(f(a), b)$ for all $a \in A$ and $b \in B$. Analogously, $\rho_B(f(a), b) \leq \rho_A(a, g(b))$ for all $a \in A$ and $b \in B$. Therefore, $\rho_A(a, g(b)) = \rho_B(f(a), b)$ for all $a \in A$ and $b \in B$. □

Corollary 1. *If a pair (f,g) is a fuzzy adjunction between $\langle A, \approx_A, \rho_A\rangle$ and $\langle B, \approx_B, \rho_B\rangle$ then (f,g) is also a fuzzy adjunction between the two fuzzy preordered sets $\langle A, \rho_A\rangle$ and $\langle B, \rho_B\rangle$.*

Conversely, if a pair (f,g) is a fuzzy adjunction between $\langle A, \rho_A\rangle$ and $\langle B, \rho_B\rangle$ then (f,g) is also a fuzzy adjunction between $\langle A, =, \rho_A\rangle$ and $\langle B, =, \rho_B\rangle$, being $=$ the standard crisp equality.

In the rest of this section, we extend the results in [12, 13] to the framework of fuzzy preordered sets wrt a fuzzy equivalence relation. The underlying idea is similar, but now the mappings f and g need to be compatible with fuzzy equivalence relations $\approx_A$ on A and $\approx_B$ on B, and this makes the development to be much more involved that in the previous case.

To begin with, it is worth to mention that the equivalences in Theorem 1 are valid when considering fuzzy equivalences: obviously, the mappings have to be compatible.

Remark 2. Given two elements $x_1, x_2 \in$ p-max(X), note that $\rho_A(x_1, x_2) = \top = \rho_A(x_2, x_1)$: on the one hand, by $x_1 \in$ p-max(X), we have that $X(x_1) = \top$ and since $x_2 \in$ p-max(X), then $X(u) \le \rho_A(u, x_2)$ for all $u \in A$. Hence, $\top = X(x_1) \le \rho_A(x_1, x_2)$ which implies that $\rho_A(x_1, x_2) = \top$.

Likewise, by $\otimes$-$\approx_A$-antisymmetry, also $(x_1 \approx_A x_2) = \top$ for $x_1, x_2 \in \approx_A$-max(X).

Theorem 3. *Let $\mathcal{A} = \langle A, \approx_A, \rho_A\rangle$ and $\mathcal{B} = \langle B, \approx_B, \rho_B\rangle$ be two fuzzy preordered sets. If the pair (f,g) is a fuzzy adjunction between $\mathcal{A}$ and $\mathcal{B}$ then $\big((f\circ g\circ f)(a) \approx_B f(a)\big) = \top$ and $\big((g \circ f \circ g)(b) \approx_A g(b)\big) = \top$, for all $a \in A, b \in B$.*

Proof. Since f is isotone and $g \circ f$ is inflationary we have

$$\top = \rho_A(a, gf(a)) \le \rho_B(f(a), fgf(a)),$$

therefore, $\rho_B(f(a), fgf(a)) = \top$.

Moreover, $\rho_B(fgf(a), f(a)) = \rho_A(gf(a), gf(a)) = \top$. Therefore, from the $\otimes$-$\approx_B$-antisymmetric property, we obtain $(f \circ g \circ f)(a) \approx_B f(a)) = \top$.

For the other composition, the proof is analogous. $\square$

Corollary 2. *Let $\mathcal{A} = \langle A, \approx_A, \rho_A\rangle$ and $\mathcal{B} = \langle B, \approx_B, \rho_B\rangle$ be two fuzzy preordered sets. If the pair (f,g) is a fuzzy adjunction between $\mathcal{A}$ and $\mathcal{B}$ then, for all $a \in A, b \in B$,*

(i) $\rho_B\big((f \circ g \circ f)(a), f(a)\big) = \rho_B\big(f(a), (f \circ g \circ f)(a)\big) = \top$
(ii) $\rho_A\big((g \circ f \circ g)(b), g(b)\big) = \rho_A\big(g(b), (g \circ f \circ g)(b)\big) = \top$.

Corollary 3. *Let $\mathcal{A} = \langle A, \approx_A, \rho_A\rangle$ and $\mathcal{B} = \langle B, \approx_B, \rho_B\rangle$ be two fuzzy preordered sets. If the pair (f,g) is a fuzzy adjunction between $\mathcal{A}$ and $\mathcal{B}$ then, for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$, the following equalities hold:*

(i) $\big(f(a_1) \approx_B f(a_2)\big) = \big((g \circ f)(a_1) \approx_A (g \circ f)(a_2)\big)$.

(ii) $\big(g(b_1) \approx_A g(b_2)\big) = \big((f \circ g)(b_1) \approx_B (f \circ g)(b_2)\big)$.

Proof. We will prove just the first item, since the second one is similar.

Given $a_1, a_2 \in A$, since g is compatible, we have that $\big(f(a_1) \approx_B f(a_2)\big) \leq \big((g \circ f)(a_1) \approx_A (g \circ f)(a_2)\big)$. On the other hand, since f is compatible, we have that

$$\big(g(f(a_1)) \approx_A g(f(a_2))\big) \leq \big(f(g(f(a_1))) \approx_B f(g(f(a_2)))\big).$$

Now, by Theorem 3, we have that $\big(f(a) \approx_B f(g(f(a)))\big) = \top$, for all $a \in A$. Finally, the $\otimes$-transitivity of $\approx_B$ leads to

$$\begin{aligned}
&\big(f(g(f(a_1))) \approx_B f(g(f(a_2)))\big) \\
&\qquad = \big(f(a_1) \approx_B f(g(f(a_1)))\big) \otimes \big(f(g(f(a_1))) \approx_B f(g(f(a_2)))\big) \\
&\qquad \leq \big(f(a_1) \approx_B f(g(f(a_2)))\big) \\
&\qquad = \big(f(a_1) \approx_B f(g(f(a_2)))\big) \otimes \big(f(g(f(a_2))) \approx_B f(a_2)\big) \\
&\qquad \leq \big(f(a_1) \approx_B f(a_2)\big)
\end{aligned}$$

□

4 Characterization and construction of the adjunction

Some more definitions are needed in order to solve the problem in the case of surjective mappings.

Definition 12. *Let $\mathcal{A} = \langle A, \approx_A, \rho_A\rangle$ and $\mathcal{B} = \langle B, \approx_B, \rho_B\rangle$ be two fuzzy preordered sets wrt $\approx_A$ and $\approx_B$, respectively, and let $f\colon \mathcal{A} \to \mathcal{B}$ be a compatible mapping. The* ***fuzzy kernel relation*** *$\equiv_f\colon A \times A \to L$ associated to f is defined as follows for $a_1, a_2 \in A$,*

$$(a_1 \equiv_f a_2) = (f(a_1) \approx_B f(a_2)).$$

Trivially, the fuzzy kernel relation is a fuzzy equivalence relation. The equivalence class of an element $a \in A$ is a fuzzy set denoted by $[a]_f\colon A \to L$ defined by $[a]_f(u) = (f(a) \approx_B f(u))$ for all $u \in A$.

The following definitions recall the notion of Hoare ordering between crisp subsets, and then introduces an alternative statement in the subsequent lemma:

Definition 13. *Let $\mathcal{A} = \langle A, \approx_A, \rho_A\rangle$ be a fuzzy preordered set wrt a fuzzy equivalence relation $\approx_A$. For C, D crisp subsets of A, consider the following notation*

- $(C \sqsubseteq_W D) = \displaystyle\bigvee_{c \in C} \bigvee_{d \in D} \rho_A(c, d)$
- $(C \sqsubseteq_H D) = \displaystyle\bigwedge_{c \in C} \bigvee_{d \in D} \rho_A(c, d)$
- $(C \sqsubseteq_S D) = \displaystyle\bigwedge_{c \in C} \bigwedge_{d \in D} \rho_A(c, d)$

Lemma 1. *Let $\mathcal{A} = \langle A, \approx_A, \rho_A \rangle$ be a fuzzy preordered set wrt a fuzzy equivalence relation $\approx_A$, $X, Y \subseteq A$ such that* $\text{p-max}(X) \neq \varnothing \neq \text{p-max}(Y)$*, then*

$$\begin{aligned}(\text{p-max}(X) \sqsubseteq_W \ \text{p-max}(Y)) &= (\text{p-max}(X) \sqsubseteq_H \ \text{p-max}(Y)) \\ &= (\text{p-max}(X) \sqsubseteq_S \text{p-max}(Y)) = \rho_A(x, y)\end{aligned}$$

for any $x \in \text{p-max}(X)$ *and* $y \in \text{p-max}(Y)$*.*

Proof. Let us show that $\rho_A(x, y) = \rho_A(\bar{x}, \bar{y})$, for any $x, \bar{x} \in \text{p-max}(X)$ and $y, \bar{y} \in \text{p-max}(Y)$: Indeed, using the transitive property of ρ_A and Remark 2 we have that

$$\rho_A(x,y) \geq \rho_A(x,\bar{x}) \otimes \rho_A(\bar{x},y) = \top \otimes \rho_A(\bar{x},y) \geq \rho_A(\bar{x},\bar{y}) \otimes \rho_A(\bar{y},y) = \rho_A(\bar{x},\bar{y}).$$

Analogously, $\rho_A(\bar{x}, \bar{y}) \geq \rho_A(x, y)$. Therefore, $\rho_A(\bar{x}, \bar{y}) = \rho_A(x, y)$ for any $x, \bar{x} \in \text{p-max}(X)$ and $y, \bar{y} \in \text{p-max}(Y)$. $\square$

Notice that, by Lemma 1, when both sets are non-empty, for any $x \in \text{p-max}(X)$ and $y \in \text{p-max}(Y)$, $\big(\text{p-max}(X) \sqsubseteq_H \ \text{p-max}(Y)\ \big) = \rho_A(x, y)$ and this justifies the following notation.

Notation 1 *Let $\mathcal{A} = \langle A, \approx_A, \rho_A \rangle$ be a fuzzy preorder wrt a fuzzy equivalence relation $\approx_A$. Let X, Y be crisp subsets of A such that* $\text{p-max}(X) \neq \varnothing \neq \text{p-max}(Y)$*, then* $\rho_A(\text{p-max}(X), \text{p-max}(Y))$ *denotes* $\big(\text{p-max}(X) \sqsubseteq_H \text{p-max}(Y)\big)$*.*

Remark 3. Let $\mathcal{A} = \langle A, \approx_A, \rho_A \rangle$ be a fuzzy preorder wrt a fuzzy equivalence relation $\approx_A$ and $X, Y \subseteq A$. Observe that for all $x_1, x_2 \in \text{p-max}(X)$ and $y_1, y_2 \in \text{p-max}(Y)$, we have that $(x_1 \approx_A y_1) = (x_2 \approx_A y_2)$:

Indeed, recall that $(x_1 \approx_A x_2) = \top = (y_1 \approx_A y_2)$, then $(x_1 \approx_A y_1) = (x_2 \approx_A x_1) \otimes (x_1 \approx_A y_1) \leq (x_2 \approx_A y_1) = (x_2 \approx_A y_1) \otimes (y_1 \approx_A y_2) \leq (x_2 \approx_A y_2)$.

Therefore, we can use the notation

$$\big(\text{p-max}(X) \approx_A \text{p-max}(Y)\big) = (x \approx_A y)$$

for any $x \in \text{p-max}(X), y \in \text{p-max}(Y)$.

Theorem 4 (Necessary conditions). *Let $\mathcal{A} = \langle A, \approx_A, \rho_A \rangle$, $\mathcal{B} = \langle B, \approx_B, \rho_B \rangle$ be two fuzzy preorders and $f \colon A \to B, g \colon B \to A$ two mappings which are compatible with the equivalence relations $\approx_A$ and $\approx_B$. If (f, g) is a fuzzy adjunction between $\mathcal{A}$ and $\mathcal{B}$ then*

1. $\text{p-max}([a]_f)$ *is non-empty for all $a \in A$.*
2. $\rho_A(a_1, a_2) \leq \rho_A(\text{p-max}([a_1]_f), \text{p-max}([a_2]_f))$*, for all $a_1, a_2 \in A$.*
3. $(a_1 \equiv_f a_2) \leq (\text{p-max}([a_1]_f) \approx_A \text{p-max}([a_2]_f))$*, for all $a_1, a_2 \in A$.*

Proof.

- *Condition* 1. We will show that $g(f(a)) \in$ p-max$([a]_f)$:
  By Theorem 3, we have $(f(a) \approx_B f(g(f(a)))) = \top$.
  On the other hand, using the $\approx_B$-reflexivity and that (f, g) is a fuzzy adjunction, for all $u \in A$,

  $$[a]_f(u) = (f(u) \approx_B f(a)) \le \rho_B(f(u), f(a)) = \rho_A(u, g(f(a))) = g(f(a)) \downarrow (u)$$

- *Condition* 2. By Theorem 1, f and g are isotone maps, thus

  $$\rho_A(a_1, a_2) \le \rho_A(g(f(a_1)), g(f(a_2)))$$

  for all $a_1, a_2 \in A$. We have just shown that $g(f(a)) \in$ p-max$([a]_f)$ for all $a \in A$, thus, from Lemma 1, we obtain that $\rho_A(a_1, a_2) \le \rho_A($p-max$([a_1]_f)$, p-max$([a_2]_f))$ for all $a_1, a_2 \in A$.
- *Condition* 3. Since g is compatible with $\approx_B$ and $\approx_A$, then $(a_1 \equiv_f a_2) = (f(a_1) \approx_B f(a_2)) \le (g(f(a_1)) \approx_A g(f(a_2)))$. But, by Condition 1, $g(f(a_i)) \in$ p-max$([a_i]_f)$.

□

Given $\mathcal{A} = \langle A, \approx_A, \rho_A\rangle$ a fuzzy preordered set wrt $\approx_A$ and a surjective mapping $f : A \to B$ compatible with $\approx_A$ and $\approx_B$, our first goal is to find sufficient conditions to define a suitable fuzzy preordering wrt $\approx_B$ on B and a mapping $g : B \to A$ compatible with $\approx_B$ and $\approx_A$ such that (f, g) is an adjoint pair.

Lemma 2. *Let $\mathcal{A} = \langle A, \approx_A, \rho_A\rangle$ be a fuzzy preorder and $\approx_B$ be a fuzzy equivalence relation on B together with a surjective mapping $f : A \to B$ compatible with $\approx_A$ and $\approx_B$. Suppose that* p-max$([a]_f) \neq \varnothing$ *for all $a \in A$. Then, $\mathcal{B} = \langle B, \approx_B, \rho_B\rangle$ is a fuzzy preorder wrt $\approx_B$, where ρ_B is the fuzzy relation defined as follows*

$$\rho_B(b_1, b_2) = \rho_A(\text{p-max}([a_1]_f), \text{p-max}([a_2]_f))$$

where $a_i \in f^{-1}(b_i)$ for each $i \in \{1, 2\}$.

Theorem 5 (Sufficient conditions). *Let $\mathcal{A} = \langle A, \approx_A, \rho_A\rangle$ be a fuzzy preorder wrt $\approx_A$ and $\approx_B$ be a fuzzy equivalence relation on B together with a surjective mapping $f : A \to B$ compatible with $\approx_A$ and $\approx_B$.*

Suppose that the following conditions hold:

1. p-max$([a]_f)$ *is non-empty for all $a \in A$.*
2. $\rho_A(a_1, a_2) \le \rho_A($p-max$([a_1]_f)$,p-max$([a_2]_f))$*, for all $a_1, a_2 \in A$.*
3. $(a_1 \equiv_f a_2) \le ($p-max$([a_1]_f) \approx_A ($p-max$([a_2]_f))$*, for all $a_1, a_2 \in A$.*

Then, there exists a mapping $g : B \to A$ compatible with $\approx_A$ and $\approx_B$ such that (f, g) is a fuzzy adjunction between the fuzzy preorders $\mathcal{A}$ and $\mathcal{B} = \langle B, \approx_B, \rho_B\rangle$, where ρ_B is the fuzzy relation introduced in Lemma 2.

Proof. Following Lemma 2, by Condition 1, there exists a fuzzy preordering ρ_B defined as follows:

$$\rho_B(b_1, b_2) = \rho_A(\text{p-max}([a_1]_f), \text{p-max}([a_2]_f))$$

where $a_i \in f^{-1}(b_i)$ for each $i \in \{1, 2\}$.

There is a number of suitable definitions of $g\colon B \to A$, and all of them can be specified as follows: given $b \in B$, we choose $g(b)$ as an element $x_b \in \text{p-max}([x]_f)$, where x is any element of $f^{-1}(b)$.

The existence of g is guaranteed by the axiom of choice, since f is surjective and for all $b \in B$ and for all $x \in f^{-1}(b)$, the set $\text{p-max}([x]_f)$ is nonempty. Moreover, $g(b)$ does not depend on the preimage of b, because $f(x) = f(y) = b$ implies $[x]_f = [y]_f$.

The compatibility of g with $\approx_B$ and $\approx_A$ follows from Condition 3:

$$(b_1 \approx_B b_2) = (f(a_1) \approx_B f(a_2)) = (a_1 \equiv_f a_2) \leq (c_1 \approx_A c_2)$$

for all $a_i \in f^{-1}(b_i)$ and $c_i \in \text{p-max}([a_i]_f)$, for $i \in \{1, 2\}$. In particular, $(b_1 \approx_B b_2) \leq (g(b_1) \approx_A g(b_2))$.

Now, due to Theorem 2, it suffices to prove that $\rho_A(a, g(b)) = \rho_B(f(a), b)$, for all $a \in A, b \in B$:

Firstly, by Lemma 1, $\rho_B(f(a), b) = \rho_A(u, v)$ for all $u \in \text{p-max}([a]_f)$ and $v \in \text{p-max}([z]_f)$ where $z \in f^{-1}(b)$. Since, by its definition, we have that $g(b) \in \text{p-max}([z]_f)$, we obtain $\rho_B(f(a), b) = \rho_A(u, g(b))$. Thus, we have to prove just that

$$\rho_A(u, g(b)) = \rho_A(a, g(b))$$

for all $u \in \text{p-max}([a]_f)$.

Given $u \in \text{p-max}([a]_f)$, we have $(f(a) \approx_B f(u)) = \top$ and $(f(a) \approx_B f(x)) \leq \rho_A(x, u)$, for all $x \in A$. In particular, $(f(a) \approx_B f(a)) \leq \rho_A(a, u)$, and then, since $\approx_A$ is reflexive, we obtain $\rho_A(a, u) = \top$. Therefore,

$$\rho_A(u, g(b)) = \rho_A(a, u) \otimes \rho_A(u, g(b)) \leq \rho_A(a, g(b))$$

On the other hand, for any $x \in f^{-1}(b)$, we have that $g(b) \in \text{p-max}([x]_f)$, then $(f(x) \approx_B f(g(b))) = \top$ which implies that $[g(b)]_f = [x]_f$, by Remark 1. Applying Condition 2,

$$\begin{aligned}\rho_A(a, g(b)) \leq \rho_A(&\text{p-max}([a]_f), \text{p-max}([g(b)]_f)) = \\ &= \rho_A(\text{p-max}([a]_f), \text{p-max}([x]_f)) = \rho_B(f(a), b).\end{aligned}$$

□

5 Conclusions

This work continues the research line initiated in [12–14] on the characterization of existence of adjunctions (and Galois connections) for mappings with unstructured codomain.

We have found necessary and sufficient conditions under which, given a fuzzy ordering ρ_A on A and a surjective mapping $f\colon \langle A, \approx_A\rangle \to \langle B, \approx_B\rangle$ compatible with respect to the fuzzy equivalences $\approx_A$ and $\approx_B$, there exists a fuzzy ordering

ρ_B and a compatible mapping $g\colon \langle B, \approx_B\rangle \to \langle A, \approx_A\rangle$ such that the pair (f, g) is a fuzzy adjunction.

As pieces of future work, on the one hand, the use of fuzzy equivalences can be taken into account in order to weaken the notion of surjective function and obtain an alternative proof based on this weaker notion. On the other hand, as stated in the introduction, considering surjective mappings is just the first step in the canonical decomposition of a general mapping $f\colon \langle A, \approx_A\rangle \to \langle B, \approx_B\rangle$, therefore we will study how to extend the obtained ordering to the whole codomain in the case that f is not surjective.

Finally, as a midterm goal, we would like to study possible links of our constructions with some recent efforts to develop a so-called theory of constructive Galois connections [7] aimed at introducing adjunctions and Galois connections within automated proof checkers.

References

1. R. Bělohlávek. Fuzzy Galois connections. *Mathematical Logic Quarterly*, 45(4):497–504, 1999.
2. R. Bělohlávek. Fuzzy relational systems: foundations and principles. Kluwer, 2002.
3. R. Bělohlávek and J. Konecny Concept lattices of isotone vs. antitone Galois connections in graded setting: Mutual reducibility revisited. *Information Sciences* 199:133–137, 2012.
4. R. Bělohlávek. and P. Osička. Triadic fuzzy Galois connections as ordinary connections. In *IEEE Intl Conf on Fuzzy Systems*, 2012.
5. U. Bodenhofer and F. Klawonn. A formal study of linearity axioms for fuzzy orderings. *Fuzzy Sets and Systems* 145(3):323–354, 2004.
6. U. Bodenhofer, B. De Baets and J. Fodor. A compendium of fuzzy weak orders: Representations and constructions. *Fuzzy Sets and Systems* 158(8):811–829, 2007.
7. D. Darais and D. Van Horn. Constructive Galois Connections: Taming the Galois Connection Framework for Mechanized Metatheory. `arXiv:1511.06965`, 2015.
8. J. Denniston, A. Melton, and S.E. Rodabaugh. Formal Contexts, Formal Concept Analysis, and Galois Connections. *Electronic Proc. in Theoretical Computer Science* 129:105–120, 2013.
9. A. Frascella. Fuzzy Galois connections under weak conditions. *Fuzzy Sets and Systems*, 172(1):33–50, 2011.
10. J. G. García, I. Mardones-Pérez, M. A. de Prada-Vicente, and D. Zhang. Fuzzy Galois connections categorically. *Math. Log. Q.*, 56(2):131–147, 2010.
11. F. García-Pardo, I.P. Cabrera, P. Cordero, and M. Ojeda-Aciego. On Galois connections and Soft Computing. *Lect. Notes in Computer Science*, 7903:224–235, 2013.
12. F. García-Pardo, I.P. Cabrera, P. Cordero, and M. Ojeda-Aciego. On the construction of fuzzy Galois connections. *Proc. of XVII Spanish Conference on Fuzzy Logic and Technology*, pages 99-102, 2014.
13. F. García-Pardo, I.P. Cabrera, P. Cordero, and M. Ojeda-Aciego. On adjunctions between fuzzy preordered sets: necessary conditions. *Lect. Notes in Computer Science*, 8536,pp. 211–221, 2014.
14. F. García-Pardo, I.P. Cabrera, P. Cordero, M. Ojeda-Aciego, and F.J. Rodríguez. On the definition of suitable orderings to generate adjunctions over an unstructured codomain. *Information Sciences* 286:173–187, 2014.

15. G. Georgescu and A. Popescu. Non-commutative fuzzy Galois connections. *Soft Computing*, 7(7):458–467, 2003.
16. J. Gibbons, F. Henglein, R. Hinze, and N. Wu. Relational Algebra by way of Adjunctions. In *Proc of Databases and Programming Languages (DBLP)*, 2015
17. J. Järvinen. Pawlak's information systems in terms of Galois connections and functional dependencies. *Fundamenta Informaticae*, 75:315–330, 2007.
18. J. Järvinen, M. Kondo, and J. Kortelainen. Logics from Galois connections. *Int. J. Approx. Reasoning*, 49(3):595–606, 2008.
19. J. Konecny. Isotone fuzzy Galois connections with hedges. *Information Sciences*, 181(10):1804–1817, 2011.
20. L. Lumpe and S.E. Schmidt. Pattern Structures and Their Morphisms. In *Proc. of Concept Lattices and Their Applications (CLA)*, CEUR Workshop Proceedings 1466:171–179, 2015
21. A. Melton, D.A. Schmidt, and G.E. Strecker. Galois connections and computer science applications. *Lect. Notes in Computer Science*, 240:299–312, 1986.
22. S.-C. Mu and J. Oliveira. Programming from Galois connections. *Journal of Logic and Algebraic Programming*, 81(6):680–704, 2012.
23. M. Nowak A Proof of Tarski's Fixed Point Theorem by Application of Galois Connections. *Studia Logica* 103(2):287–301, 2014
24. Y. Shi, M. Nachtegael, D. Ruan, and E. Kerre Fuzzy adjunctions and fuzzy morphological operations based on implications. *Intl J of Intelligent Systems* 24(12):1280–1296, 2009.
25. F.J. Valverde-Albacete, C. Peláez-Moreno, and C. del Campo. Activating Generalized Fuzzy Implications from Galois Connections. In *Enric Trillas: A passion for fuzzy sets*, Studies in Fuzzy Sets and Soft Computing 322:201–212, 2015.
26. M. Wolski. Galois connections and data analysis. *Fundamenta Informaticae*, 60:401–415, 2004.
27. W. Yao and L.-X. Lu. Fuzzy Galois connections on fuzzy posets. *Mathematical Logic Quarterly 55:105–112, 2009*