Finding p-indecomposable Functions: FCA Approach

Artem Revenko[1,2]

[1] TU Wien
Karlsplatz 13, 1040 Vienna, Austria
[2] TU Dresden
Zellescher Weg 12-14, 01069 Dresden, Germany

Abstract. The parametric expressibility of functions is a generalization of the expressibility via composition. All parametrically closed classes of functions (p-clones) form a lattice. For finite domains the lattice is shown to be finite, however straight-forward iteration over all functions is infeasible, and so far the p-indecomposable functions are only known for domains with two and three elements. In this work we show how p-indecomposable functions can be computed more efficiently by means of an extended version of attribute exploration (AE). Due to the growing number of attributes standard AE is not able to guarantee the discovery of all p-indecomposable functions. We introduce an extension of AE and investigate its properties. We investigate the conditions allowing us to guarantee the success of exploration. In experiments the lattice of p-clones on three-valued domain was reconstructed.

Keywords: parametric expressibility, attribute exploration, p-indecomposable function

1 Introduction

The expressibility of functions is a major topic in mathematics and has a long history of investigation. The interest is explainable: when one aims at investigating any kind of functional properties, which classes of functions should one consider? If a function f is expressible through a function h then it often means that f inherits properties of h and should not be treated separately. Moreover, if h in turn is expressible through f then both have similar or even the same properties. Therefore, partition with respect to expressibility is meaningful and can be the first step in the investigation of functions.

With the development of electronics and logical circuits a new question arises: if one wants to be able to express all possible functions which minimal set of functions should one have at hands? One of the first investigations in this direction was carried out in [Pos42]; in this work all the Boolean classes of functions closed under expressibility are found and described. Afterwards many important works were dedicated to related problems such as the investigation of the structure of the lattice of functional classes, for example, [Yab60,Ros70]. However, it

© Sadok Ben Yahia, Jan Konecny (Eds.): CLA 2015, pp. 35–46,
ISBN 978–2–9544948–0–7, ISSN 2311–701X, Blaise Pascal University, LIMOS laboratory, Clermont-Ferrand, 2015.

is known that the lattice of classes of functions closed under expressibility is in general uncountably infinite. In [Kuz79] a more general type of functional expressibility was introduced – parametric expressibility. A significant advantage of this type of expressibility is that for any finite domain $A_k, |A|= k$ the lattice of all classes closed under parametric expressibility classes of functions (p-clones) is finite [BW87]. However, finding this lattice is a complex task. For $k = 3$ in a thorough and tedious investigation [Dan77] it was proved that a system of 197 functions forms the lattice of all p-clones. The investigation was carried out without the use of computers.

In this paper we introduce, develop, and investigate the methods and tools for automation of the exploration of the lattice of p-clones. Therefore, this paper "applied" to A_3 can be seen as complementing the work [Dan77] where a proof of the correctness of the results obtained using the elaborated in this paper tools can be found. Namely, in this paper we answer the question **how** to find all the p-clones, whereas in [Dan77] it is proved that certain functions allow us to construct the desired lattice. The presented methods and tools are extensible to larger domains as well.

Contributions

- New original approach to exploring the lattice of p-clones introduced;
- An extension of the standard exploration procedure is introduced and investigated;
- The whole procedure is implemented and executed; the obtained results confirm with the previously known results;
- It is proved that for certain starting conditions the desired lattice will necessarily be eventually discovered.

2 Formal Concept Analysis

In what follows we keep to standard definitions of FCA [GW99]. Let G and M be sets and let $I \subseteq G \times M$ be a binary relation between G and M. The triple $\mathbb{K} := (G, M, I)$ is called a *(formal) context*. The set G is called the set of *objects*. The set M is called the set of *attributes*. A context (G_*, M_*, I_*) such that $G_* \subseteq G,\ M_* \subseteq M$, and $I_* = I \cap G_* \times M_*$ is called a *subcontext* of $\mathbb{K}$.

Consider mappings $\varphi\colon 2^G \to 2^M$ and $\psi\colon 2^M \to 2^G$:

$$\varphi(X) := \{m \in M \mid gIm \text{ for all } g \in X\},$$

$$\psi(A) := \{g \in G \mid gIm \text{ for all } m \in A\}.$$

Mappings φ and ψ define a *Galois connection* between $(2^G, \subseteq)$ and $(2^M, \subseteq)$, i.e. $\varphi(X) \subseteq A \Leftrightarrow \psi(A) \subseteq X$. Usually, instead of φ and ψ a single notation $(\cdot)'$ is used.

Let $X \subseteq G,\ A \subseteq M$. A *formal concept* C of a formal context (G, M, I) is a pair (X, A) such that $X' = A$ and $A' = X$. The subset of objects X is called the *extent of* C and is denoted by $\text{ext}(C)$, and the subset of attributes A is called the *intent of* C and is denoted by $\text{int}(C)$. For a context (G, M, I), a concept $C_1 = (X, A)$ is a *subconcept* of a concept $C_2 = (Y, B)$ $(C_1 \leq C_2)$ if $X \subseteq Y$ or, equivalently, $B \subseteq A$. This defines a partial order on formal concepts. The set of all formal concepts of (G, M, I) is denoted by $\mathfrak{B}(G, M, I)$.

An *implication* of $\mathbb{K} = (G, M, I)$ is defined as a pair (A, B), where $A, B \subseteq M$, written $A \to B$. A is called the *premise*, B is called the *conclusion* of the implication $A \to B$. The implication $A \to B$ is *respected by a set of attributes* N if $A \nsubseteq N$ or $B \subseteq N$. We say that the implication is *respected by an object* g if it is respected by the intent of g. If g does not respect an implication then g is called a *counter-example*. The implication $A \to B$ *holds* (is *valid*) in $\mathbb{K}$ if it is respected by all g', $g \in G$, i.e. every object, that has all the attributes from A, also has all the attributes from B $(A' \subseteq B')$. A *unit implication* is defined as an implication with only one attribute in its conclusion, i.e. $A \to b$, where $A \subseteq M,\ b \in M$. Every implication $A \to B$ can be regarded as a set of unit implications $\{A \to b \mid b \in B\}$.

An *implication basis* of a context $\mathbb{K}$ is defined as a set $\mathfrak{L}_{\mathbb{K}}$ of implications of $\mathbb{K}$, from which any valid implication for $\mathbb{K}$ can be obtained as a consequence and none of the proper subsets of $\mathfrak{L}_{\mathbb{K}}$ has this property. We call the set of all valid in $\mathbb{K}$ the *implicative theory* of $\mathbb{K}$. A minimal in the number of implications basis was defined in [GD86] and is known as the *canonical implication basis*.

An object g is called *reducible* in a context $\mathbb{K} := (G, M, I)$ iff $\exists X \subseteq G \setminus g : g' = X'$. Note that a new object is going to be reducible if in the context there already exists a formal concept with the same intent as the intent of the new object. Reducible objects neither contribute to any implication basis nor to the concept lattice [GW99], therefore, if one is only interested in the implicative theory or in the concept lattice of the context reducible objects can be eliminated. In what follows we introduce other types of reducibility, therefore, we refer to this type of reducibility as *plain* reducibility.

In what follows the canonical implication basis is used, however, the investigation could be performed using another implication basis.

Attribute Exploration (AE) consists in iterations of the following steps until stabilization: computing the implication basis of a context, finding counterexamples to implications, updating the context with counterexamples as new objects, recomputing the basis. AE has been successfully used for investigations in many mostly analytical areas of research. For example, in [KPR06] AE is used for studying Boolean algebras, in [Dau00] lattice properties are studied, in [Rev14] algebraic identities are studied.

3 Expressibility of Functions

Consider a set A_k, $|A| = k, k \in \mathbb{N}$. Consider a function $f : A^{ar(f)} \to A$ ($ar(f)$ denotes the arity of f), the set of all possible functions over A_k of different arities is denoted by U_k. The particular functions $p_n^i(x_1, \ldots x_n) = x_i$ are called the *projections*. The set of all projections is denoted by Pr. In what follows instead of writing $(x_1, \dots x_n)$ we use a shorter notation $(\mathbf{x})$.

Let $H \subseteq U_k$. We say that f is *compositionally expressible* through H (denoted $f \leq H$) if the following condition holds:

$$f(\mathbf{x}) \quad \equiv \quad h(j_1(\mathbf{x}), \dots, j_{ar(h)}(\mathbf{x})), \tag{1}$$

for some $h, j_1, \dots j_m \in H \cup Pr$.

A *functional clone* is a set of functions containing all projections and closed under compositions. The set of all functional clones over a domain of size $k = 2$ forms a countably infinite lattice [Pos42]. However, if $k > 2$ then the set of all functional classes is uncountable [YM59].

Let $H \subseteq U_k$ and for any $i \in [1, m]: \ t_i, s_i \in H \cup Pr$. We say that $f, \ f \in U_k$ is *parametrically expressible* through H (denoted $f \leq_p H$) if the following condition holds:

$$f(\mathbf{x}) = y \quad \Longleftrightarrow \quad \exists \mathbf{w} \bigwedge_{i=1}^{m} t_i(\mathbf{x}, \mathbf{w}, y) = s_i(\mathbf{x}, \mathbf{w}, y). \tag{2}$$

The notation $J \leq_p H$ means that every function from J is parametrically expressible through H. A *parametric clone* (or *p-clone*) is a set of functions closed under parametric expressibility and containing all projections. We consider a special relation $f^\bullet$ of arity $ar(f)+1$ on A_k called the *graph* of function f. $f^\bullet$ consists of the tuples of the form $(\mathbf{x}, f(\mathbf{x}))$. If function h is compatible with $f^\bullet$, i.e. if for all valuations of variables x_{ij} in A_k holds the identity $(ar(f) = n, \ ar(h) = m)$

$$f(h(x_{11}, \dots, x_{1m}), \dots h(x_{n1}, \dots, x_{nm})) \equiv h(f(x_{11}, \dots, x_{n1}), \dots f(x_{1m}, \dots, x_{nm})),$$

then we say that functions f and h *commute* (denoted $f \perp h$). For a set of functions H we write $f \perp H$ to denote that for all $h \in H: \ f \perp h$. The commutation property is commutative, i.e. $f \perp h$ iff $h \perp f$.

The *centralizer* of H is defined by $H^\perp = \{g \in U_k \mid g \perp H\}$. In [Kuz79] it is shown that if $f \leq_p H$ then $f \perp H^\perp$.

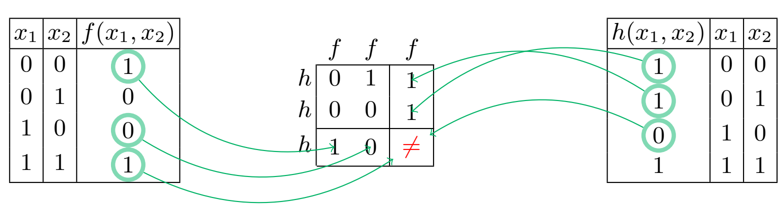

Fig. 1. Functions f and h do not commute

A function f is called *p-indecomposable* if each system H parametrically equivalent to $\{f\}$ (i.e. $f \leq_p H$ and $H \leq_p f$) contains a function parametrically equivalent to f. Hence, for each p-indecomposable function there exists a class of p-indecomposable functions that are parametrically equivalent to it. From each such class we take only one representative (only one p-indecomposable function) and gather them in a set of p-indecomposable functions denoted by F_k^p. A p-clone H cannot be represented as an intersection of p-clones strictly containing H if and only if there exists a p-indecomposable function f such that $H = f^{\perp\perp}$. Hence, in order to construct the lattice of all p-clones it suffices to find all p-indecomposable functions. The lattice of all p-clones for any finite k is finite [BW87], hence, F_k^p is finite.

In [BW87] it is proved that it suffices to consider p-indecomposable functions of arity at most k^k, however, the authors conjecture that the actual arity should be equal to k for $k \geq 3$. The conjecture is still open. Nevertheless, thanks to results reported in [Dan77], we know that the conjecture holds for $k = 3$.

4 Exploration of P-clones

The knowledge about the commutation properties of a finite set of functions $F \subseteq U_k$ can be represented as a formal context $\mathbb{K}_F = (F, F, \perp_F)$, where $\perp_F \subseteq F^2$, a pair $(f_1, f_2) \in F^2$ belongs to the relation $\perp_F$ iff $f_1 \perp f_2$. Note that the relation $\perp_F$ is symmetric, hence, the objects and the attributes of the context are the same functions.

The goal of this paper is to develop methods for constructing the lattice of all p-clones on A_3. As already noted, for the purpose of constructing the lattice of p-clones it suffices to find all p-indecomposable functions F_k^p. The set of supremum-irreducible elements of the lattice of p-clones is exactly the set $\{f^{**} \mid f \in F_k^p\}$.

For any domain of size k there exist k^{k^k} functions of arity k. Therefore, to compute the context of all commuting functions $\mathbb{K}_{U_k}$ one has to perform $O(k^{k^k} * k^{k^k} * k^{k^2})$ operations (taking into consideration only functions of arity k and the cost of commutation check in the worst case). For $k = 3$ we count about 10^{30} operations. Therefore, already for $k = 3$ a brute-force solution is infeasible.[3]

We intend to apply AE to commuting functions. For this purpose we developed and implemented methods for finding counter-examples to implications over functions from U_k [Rev15]. These methods are not presented in this paper for the sake of compactness. However, as the number of attributes is not fixed, the success of applying AE is not guaranteed, i.e. it is not guaranteed that the complete lattice of p-clones will eventually be discovered using AE.

4.1 Object-Attribute Exploration

We now describe which commuting properties a new function $g \notin F$ should possess in order to alter the concept lattice of the original context $\mathbb{K} = (F, F, \perp)$ despite the fact that the intent of g is equal to an intent from $\mathfrak{B}(F, F, \perp_F)$.

[3] Of course one can use dualities, but it does not give a feasible solution as well as there exist only $k * (k - 1)$ dualities.

To distinguish between binary relations on different sets of functions we use subscripts. The commutation relation on F is denoted by $\perp_F$, i.e. $\perp_F = \{(h,j) \in F^2 \mid h \perp j\}$. The context with the new function $(F \cup g, F \cup g, \perp_{F\cup g})$ is denoted by $\mathbb{K}_{F\cup g}$. The derivation operator for the context $\mathbb{K}_{F\cup g}$ is denoted by $(\cdot)^{\perp_{F\cup g}}$.

Proposition 1. *Let $C \in \mathfrak{B}(F,F,\perp)$ such that ext$(C) \nsubseteq$ int(C). Let $g \in U_k, g \notin F$ be a function such that $g^{\perp_{F\cup g}} \cap F =$ int(C) (g is reducible in $\mathbb{K}_F$).*
g is irreducible in $\mathbb{K}_{F\cup g} \quad \Leftrightarrow \quad g \perp g$.

Proof. As $\text{ext}(C) \nsubseteq \text{int}(C)$ and for all $f \in F \setminus \text{int}(C) : \ g \not\perp f$ it follows that $g \not\perp \text{ext}(C)$. We prove the contrapositive statement: g is reducible in $\mathbb{K}_{F\cup g} \quad \Leftrightarrow g \not\perp g$.

- $\Leftarrow$ As $g \not\perp g$ we have $g^{\perp_{F\cup g}} = \text{int}(C) = \text{ext}(C)^{\perp_{F\cup g}}$. Therefore, g is reducible.
- $\Rightarrow$ As g is reducible we obtain $g^{\perp_{F\cup g}} = H^{\perp_{F\cup g}}$ for some $H \subseteq F$. Fix this H. As $H^{\perp_{F\cup g}} = \text{int}(C)$ we have $H^{\perp_{F\cup g}\perp_{F\cup g}} = \text{ext}(C)$. Suppose $H \subseteq \text{int}(C)$, then $H^{\perp_{F\cup g}\perp_{F\cup g}} \subseteq \text{int}(C)^{\perp_{F\cup g}\perp_{F\cup g}} = \text{int}(C)$. As $H^{\perp_{F\cup g}\perp_{F\cup g}} = \text{ext}(C)$ and $\text{ext}(C) \nsubseteq \text{int}(C)$ we arrive at a contradiction. Therefore, $H \nsubseteq \text{int}(C)$. Hence, $g \not\perp H$, therefore, $g \notin H^{\perp_{F\cup g}}$, hence, $g \notin g^{\perp_{F\cup g}}$.

Corollary 1. *If g is reducible in $\mathbb{K}_F$, but irreducible in $\mathbb{K}_{F\cup g}$ and $g \perp g$ then ext$(C) \to g$ holds in $\mathbb{K}_{F\cup g}$.*

Proof. As $g^{\perp_{F\cup g}} = \text{int}(C) \cup \{g\}$ and $\text{ext}(C)^{\perp_{F\cup g}} = \text{int}(C)$ we have $\text{ext}(C)^{\perp_{F\cup g}} \subset g^{\perp_{F\cup g}}$, therefore, $\text{ext}(C) \to g$.

The statement dual to Proposition 1 holds as well.

Proposition 2. *Let $C \in \mathfrak{B}(F,F,\perp_F)$ such that ext$(C) \subseteq$ int(C). Let $g \in U_k, g \notin F$ be a function such that $g^{\perp_{F\cup g}} \cap F =$ int(C) (g is reducible in $\mathbb{K}_F$).*
g is irreducible in $\mathbb{K}_{F\cup g} \quad \Leftrightarrow \quad g \not\perp g$.

Proof. As $\text{ext}(C) \subseteq \text{int}(C)$ and $g \perp \text{int}(C)$ then $g \perp \text{ext}(C)$. We prove the contrapositive statement: g is reducible in $\mathbb{K}_{F\cup g} \quad \Leftrightarrow \quad g \perp g$.

- $\Leftarrow$ As $g \perp g$ and $g \perp \text{ext}(C)$ we have $\text{ext}(C)^{\perp_{F\cup g}} = \text{int}(C) \cup \{g\} = g^{\perp_{F\cup g}}$. Hence, g is reducible.
- $\Rightarrow$ As g is reducible we obtain $g^{\perp_{F\cup g}} = H^{\perp_{F\cup g}}$ for some $H \subseteq F$. Fix this H. As $g \perp \text{int}(C)$ we have $H \perp \text{int}(C)$, hence, $H \subseteq \text{ext}(C)$. As $g \perp \text{ext}(C)$ we have $g \perp H$, hence, $g \in H^{\perp_{F\cup g}}$, therefore, $g \in g^{\perp_{F\cup g}}$ and $g \perp g$.

Corollary 2. *If g is reducible in $\mathbb{K}_F$, but irreducible in $\mathbb{K}_{F\cup g}$ and $g \not\perp g$ then $g \to$ ext(C) holds in $\mathbb{K}_{F\cup g}$.*

Proof. As $g^{\perp_{F\cup g}} = \text{int}(C)$ and $\text{ext}(C)^{\perp_{F\cup g}} = \text{int}(C) \cup \{g\}$ we have $g^{\perp_{F\cup g}} \subset \text{ext}(C)^{\perp_{F\cup g}}$, therefore, $g \to \text{ext}(C)$.

In order to distinguish reducibility in the old context $\mathbb{K}_F$ and in the new context $\mathbb{K}_{F\cup g}$ we introduce a new notation.

Definition 1. *We call a function g that is reducible in $\mathbb{K}_F$, but irreducible in $\mathbb{K}_{F\cup g}$,* first-order irreducible for $\mathbb{K}_F$. *If g is reducible for $\mathbb{K}_F$ and reducible in $\mathbb{K}_{F\cup g}$ we call it* first-order reducible for $\mathbb{K}_F$.

We remind that if g is irreducible in $(F\cup g, F, \perp_F \cup\{(g,f) \in \{g\}\times F \mid f \perp g\})$ we call it plainly irreducible. Hence, if function is first-order reducible for $\mathbb{K}_F$ then it is also plainly reducible in $\mathbb{K}_F$. Note that g is plainly irreducible in $\mathbb{K}_F$ iff g is a counter-example to some valid in $\mathbb{K}_F$ implication.

Next we present an example with functions from U_3, in order to explicitly show this we add 3 in the subscript of every function. The numbering of the functions is induced by the lexicographic ordering on the outputs of the functions [Rev15]. We use superscripts $\cdot^u$ for unary, $\cdot^b$ for binary, and $\cdot^t$ for ternary functions.

Example 1. The context under consideration $\mathbb{K}_0^{(3)}$ is presented in Figure 2. The implication basis of $\mathbb{K}_0^{(3)}$ is empty, therefore, there exist no plainly irreducible functions. The function $f^b_{3,756}$ has the following commuting properties: $f^b_{3,756} \perp \{f^u_{3,0}, f^b_{3,12015}\}$ and $f^b_{3,756} \not\perp f^u_{3,1}$. Moreover, $f^b_{3,756} \not\perp f^b_{3,756}$ and for the corresponding concept C holds $\text{ext}(C) = \{f^u_{3,0}\} \subset \{f^u_{3,0}, f^b_{3,12015}\} = \text{int}(C)$. As follows from Proposition 2, the function $f^b_{3,756}$ is first-order irreducible for $\mathbb{K}_0^{(3)}$.

	$f^u_{3,0}$	$f^u_{3,1}$	$f^b_{3,12015}$
$f^u_{3,0}$	×		×
$f^u_{3,1}$		×	×
$f^b_{3,12015}$	×	×	

Fig. 2. Context $\mathbb{K}_0^{(3)}$ of functions on domain A_3 containing $f^u_{3,0}, f^u_{3,1}, f^b_{3,12015}$

Corollary 3. *Let $C \in \mathfrak{B}(F, F, \perp_F)$, $g \in U_k$, $g \notin F$, and g be first-order reducible for $\mathbb{K}_F$.*

$$ext(C) \perp g \quad \Leftrightarrow \quad g \perp g.$$

Proof. Follows from Propositions 1 and 2 and the fact that $\text{ext}(C) \perp g \Leftrightarrow \text{ext}(C) \subseteq \text{int}(C)$.

There remains a possibility that a union of sets of reducible functions is irreducible. We proceed with the simplest case when there are only two sets each containing a single first-order reducible function for the current context. We prove several propositions about such pairs of first-order reducible functions. The consequences of these propositions are deeper investigated in Section 4.2.

We consider a context $\mathbb{K}_F$ and new functions $g_1, g_2 \in U_k$, $g_1, g_2 \notin F$. We denote $\{g_1, g_2\}$ by G, $\perp_{F\cup G} = \{(h,j) \in (F \cup G)^2 \mid h \perp j\}$, the context $(F \cup$

$G, F \cup G, \perp_{F \cup G})$ is denoted by $\mathbb{K}_{F \cup G}$, the corresponding derivation operator is denoted by $(\cdot)^{\perp_{F \cup G}}$. As in the case with one function, for $i \in \{1,2\} : g_i$ is not a counter-examples to a valid implication iff $g_i^{\perp_{F \cup G}} \cap F \in \text{int}(G, M, I)$. We denote the corresponding intents by $\text{int}(C_1)$ and $\text{int}(C_2)$, respectively.

Proposition 3. *Let $C_1, C_2 \in \mathfrak{B}(F, F, \perp_F)$ and $g_1, g_2 \notin F$ be first-order reducible for $\mathbb{K}_F$. Suppose $g_1 \perp g_2$.*
Both g_1, g_2 are irreducible in $\mathbb{K}_{F \cup G}$ $\Leftrightarrow$ ext$(C_1) \nsubseteq$ int(C_2).

Proof. As g_1 is irreducible it holds that $g_1^{\perp_{F \cup G}} \neq \text{ext}(C_1)^{\perp_{F \cup G}}$. From Corollary 3 follows that $g_1 \in \text{ext}(C_1)^{\perp_{F \cup G}}$ iff $g_1 \in g_1^{\perp_{F \cup G}}$. Therefore, $\text{ext}(C_1)^{\perp_{F \cup G}} = g_1^{\perp_{F \cup G}} \setminus \{g_2\}$. Hence, $\text{ext}(C_1) \not\perp g_2$, hence, $\text{ext}(C_1) \nsubseteq \text{int}(C_2)$. Similarly for g_2, $\text{ext}(C_2) \nsubseteq \text{int}(C_1)$.

Proposition 4. *Let $C_1, C_2 \in \mathfrak{B}(F, F, \perp_F)$ and $g_1, g_2 \notin F$ be first-order reducible for $\mathbb{K}_F$. Suppose $g_1 \not\perp g_2$.*
Both g_1, g_2 are irreducible in $\mathbb{K}_{F \cup G}$ $\Leftrightarrow$ ext$(C_1) \subseteq$ int(C_2).

Proof. As g_1 is irreducible it holds that $g_1^{\perp_{F \cup G}} \neq \text{ext}(C_1)^{\perp_{F \cup G}}$. From Corollary 3 follows that $g_1 \in \text{ext}(C_1)^{\perp_{F \cup G}}$ iff $g_1 \in g_1^{\perp_{F \cup G}}$. Therefore, $\text{ext}(C_1)^{\perp_{F \cup G}} = g_1^{\perp_{F \cup G}} \cup \{g_2\}$. Hence, $\text{ext}(C_1) \perp g_2$, hence, $\text{ext}(C_1) \subseteq \text{int}(C_2)$. By the properties of derivation operators, $\text{ext}(C_2) \subseteq \text{int}(C_1)$.

The functions mentioned in Propositions 4 and 3 can be called *second-order irreducible for* $\mathbb{K}_F$. In the next proposition we show that it is not necessary to look for three functions at once in order to find all p-indecomposable functions. Therefore, we do not need to define third-order irreducibility.

Here we use the notation: for $I \subseteq \{1,2,3\} : \; L_I = \{g_i \mid i \in I\}$. We omit the curly brackets in I, i.e. $L_{\{1,2\}} = L_{12} = \{g_1, g_2\}$.

Proposition 5. *Let $G = \{g_1, g_2, g_3\}$ be a set of functions such that $G \cap F = \emptyset$ and for $i \in \{1,2,3\} : \; g_i^{\perp_{F \cup G}} \cap F = $ int(C_i). If not all functions from G are reducible in $\mathbb{K}_{F \cup G}$ then there exists $L \subset G$ such that not all functions from L are reducible in $\mathbb{K}_{F \cup L}$.*

Proof. Let g_1 be reducible in $\mathbb{K}_{F \cup L_{12}}$ and in $\mathbb{K}_{F \cup L_{13}}$. Then there exists $H \subseteq F \cup \{g_2\} : \; H^{\perp_{F \cup L_{12}}} = g_1^{\perp_{F \cup L_{12}}}$ and $J \subseteq F \cup \{g_3\} : \; J^{\perp_{F \cup L_{13}}} = g_1^{\perp_{F \cup L_{13}}}$. Fix these H and J. If either g_2 is irreducible in $\mathbb{K}_{F \cup L_2}$ or g_3 is irreducible in $\mathbb{K}_{F \cup L_3}$ then the proposition is proved. Therefore, we can assume that they are reducible in corresponding context. Hence, without loss of generality, we can assume that $H, J \subseteq F$ (i.e. $H \cap G = J \cap G = \emptyset$). Note that

$$g_1^{\perp_{F \cup G}} = g_1^{\perp_{F \cup L_{13}}} \cup g_1^{\perp_{F \cup L_{12}}} = J^{\perp_{F \cup L_{13}}} \cup H^{\perp_{F \cup L_{12}}}. \tag{3}$$

Let $g_3 \in H^{\perp_{F \cup G}}$. Then $g_3 \perp H$. As $g_3^{\perp_{F \cup G}} \cap F = \text{int}(C_3)$ we obtain $H \subseteq \text{int}(C_3)$. Moreover, as $\text{int}(C_3)$ is an intent in $\mathbb{K}_F$ we have $H^{\perp_F \perp_F} \subseteq \text{int}(C_3)$. As $g_1^{\perp_{F \cup G}} \cap F = H^{\perp_F} = J^{\perp_F} = \text{int}(C_1)$ we have $J^{\perp_F \perp_F} \subseteq \text{int}(C_3)$ and, by properties of closure operators, $J \subseteq \text{int}(C_3)$. Therefore, $g_3 \perp J$ and $g_3 \in J^{\perp_{F \cup G}}$. Similarly, if $g_2 \in J^{\perp_{F \cup G}}$ then $g_2 \in H^{\perp_{F \cup G}}$. Hence,

$$H^{\perp_{F \cup L_{12}}} \cup J^{\perp_{F \cup L_{13}}} = H^{\perp_{F \cup G}} \cup J^{\perp_{F \cup G}}. \tag{4}$$

Combining (3) and (4) we obtain $g_1^{\perp_{F \cup G}} = H^{\perp_{F \cup G}} \cup J^{\perp_{F \cup G}}$. Therefore, $g_1^{\perp_{F \cup G}} = (H \cap J)^{\perp_{F \cup G}}$. Hence, g_1 is reducible in $\mathbb{K}_{F \cup G}$ and we arrive at a contradiction with initial assumption.

Therefore, if g_1, g_2 are in $\mathbb{K}_{F \cup L_{12}}$ then at least g_1 is irreducible in $\mathbb{K}_{F \cup L_{13}}$. If g_3 is reducible in $\mathbb{K}_{F \cup L_{13}}$ then g_1 is reducible in $\mathbb{K}_{F \cup L_1}$. Otherwise, both g_1, g_3 are irreducible in $\mathbb{K}_{F \cup L_{13}}$.

Suppose that a context $\mathbb{K}_F$ contains all p-indecomposable functions, however, the task is to prove this fact, i.e. that no further p-indecomposable functions exist. Suppose it has been checked that no counter-examples exist and every single function $g \in U_k$ is first-order reducible for $\mathbb{K}_F$. According to the above propositions it is necessary to look for exactly two functions at once in order to prove the desired statement. Therefore, in order to complete the proof for every $C_1, C_2 \in \mathfrak{B}(\mathbb{K}_F)$ one has to find all the functions g_1, g_2 such that $g_1^{\perp_{F \cup g_1}} \cap F = \text{int}(C_1)$ and $g_2^{\perp_{F \cup g_2}} \cap F = \text{int}(C_2)$ and then check if g_1 commutes with g_2. Therefore, one has to check the commutation property between all functions (if the context indeed contains all p-indecomposable functions). As already discussed, this task is infeasible. This result is discouraging. However, having the knowledge about the final result in some cases we can guarantee that all p-indecomposable functions will be found even without looking for two functions at once.

4.2 Implicatively Closed Subcontexts

During the exploration of p-clones one can discover such a subcontext of functions that no further function is a counter-example to existing implications. We shall say that such a subcontext is *implicatively closed*, meaning that all the valid in this subcontext implications are valid in the final context as well. Analysis of similar constructions can be found in [Gan07].

In order to guarantee the discovery of all p-indecomposable functions (success of exploration) it would suffice to find such a subcontext that it is neither implicatively closed nor contained in any other implicatively closed subcontext. Suppose the context $\mathbb{K}_F = (F, F, \perp_F),\ F \subseteq U_k$ is discovered. As earlier, we denote the context of all p-indecomposable functions on U_k by $\mathbb{K}_{F_k^p}$. Let $S = F_k^p \backslash F$. It would be desirable to be able to guarantee the discovery of functions S by considering only the discovered part of relation $\perp_F$ and the part $\perp_{FS}$ ($=\perp_{SF}^{-1}$), see Figure 3. Unfortunately, as the next example shows, in general it is not possible.

Example 2. Consider the context in Figure 4. The context contains all the p-indecomposable functions from U_2 and three additional objects g_1, g_2, g_3. Functions with commutation properties as of g_1, g_2, g_3 do not exist. However, if functions with commutation properties as of g_1, g_2, g_3 existed then the functions g_1, g_2

	F	S
F	$\bot_F$	$\bot_{FS}$
S	$\bot_{SF}$	$\bot_S$

Fig. 3. Partitioning of the context $\mathbb{K}_{F_k^p}$ of all p-indecomposable functions

would not be counter-examples to any valid in $\mathbb{K}_{F_2^p \cup g_3}$ implication. Note that g_3 is a counter-example to a valid in $\mathbb{K}_{F_2^p}$ implication. Therefore, the subcontext containing functions $F_2^p \cup g_3$ would be implicatively closed. Moreover, it is even closed with respect to finding first-order irreducible functions as g_1 is reducible in $\mathbb{K}_{F_2^p \cup \{g_1, g_3\}}$ and g_2 is reducible in $\mathbb{K}_{F_2^p \cup \{g_2, g_3\}}$.

However, if instead of g_3 we consider the function g_4, which differs from g_3 only in that g_4 commutes with both g_1 and g_2, then the subcontext containing $F_2^p \cup g_4$ is neither implicatively closed nor contained in any implicatively closed subcontext of the context $\mathbb{K}_{F_2^p \cup \{g_1, g_2, g_4\}}$. The difference between g_3 and g_4 is contained in $\bot_S$ in Figure 3. Therefore, in general it is not possible to guarantee the discovery of functions S without considering $\bot_S$.

	f_0^u	f_1^u	f_{14}^b	f_8^b	f_{212}^t	f_{150}^t	f_3^u	g_3	g_4	g_1	g_2
f_0^u	×		×	×	×	×		×	×	×	
f_1^u		×			×	×					
f_{14}^b	×		×				×	×	×		
f_8^b	×			×			×				×
f_{212}^t	×	×					×	×	×		
f_{150}^t	×	×				×	×				
f_3^u			×	×	×	×	×	×	×	×	×
g_3	×		×		×		×				
g_4	×		×		×		×			×	×
g_1	×						×		×		×
g_2				×			×		×	×	×

Fig. 4. Context $\mathbb{K}_{F_2^p \cup \{g_1, g_2, g_3\}}$ from Example 2

Definition 2. *Let* $\mathbb{K}_H$ *be a context,* $\mathbb{K}_F \subseteq \mathbb{K}_H$*,* $S = H \setminus F$*. An object* $s \in S$ *is called an* essential counter-example *for* $\mathbb{K}_F$ *if there exists a valid in* $\mathbb{K}_F$ *implication* Imp *such that*

1. *s is a counter-example to Imp;*
2. *there does not exist an object $p \in S \setminus \{s\}$ such that p is a counter-example to Imp.*

It is clear that all the essential counter-examples will necessarily be added to the context during the exploration. The next proposition suggests how one can check if a counter-example is essential or not.

In the context $\mathbb{K}_{F_3^p}$ there are several pairs of functions (f_1, f_2) such that they commute with the same functions except for one commutes with itself and the other does not commute with itself. These functions cannot be essential counter-examples, because they are counter-examples to the same implications, if any. However, if they are the only counter-examples to some valid implication then these functions will eventually be discovered by object-attribute exploration.

Proposition 6. *Let $s_1, s_2 \in S$ such that $s_2 \not\perp s_2$ and $s_1^{\perp_{U_k}} = s_2^{\perp_{U_k}} \cup \{s_2\}$. If there exists a valid in $\mathbb{K}_F$ implication Imp such that the counter-examples are exactly $s_1, s_2 \in S$ then s_1 is first-order irreducible for $\mathbb{K}_{F \cup s_2}$ and s_2 is first-order irreducible for $\mathbb{K}_{F \cup s_1}$.*

Proof. s_1 **in** $\mathbb{K}_{F \cup s_2}$**.** As Imp is valid in $\mathbb{K}_F$ the set $s_2^{\perp_{F \cup s_1}}$ is closed in $\mathbb{K}_F$. Therefore, as follows from Proposition 1 for the object concept of s_2 ($\text{ext}(C_{s_2}) \nsubseteq \text{int}(C_{s_2})$), the function s_1 ($s_1 \perp s_1$) is first-order irreducible.

s_2 **in** $\mathbb{K}_{F \cup s_1}$**.** As Imp is valid in $\mathbb{K}_F$ the set $s_1^{\perp_{F \cup s_2}}$ is closed in $\mathbb{K}_F$. Therefore, as follows from Proposition 2 for the object concept of s_1 ($\text{ext}(C_{s_1}) \subseteq \text{int}(C_{s_1})$), the function s_2 ($s_2 \not\perp s_2$) is first-order irreducible.

We have investigated different types of reducibilities, we have shown, that there do not exist third-order irreducible functions. However, the task of finding second-order irreducible functions is infeasible. Fortunately, it is possible to find not only zero-order irreducible functions, but also first-order irreducible functions. Moreover, if it would be possible to prove that the functions undiscovered at the moment are not second-order irreducible then we can guarantee that all the p-indecomposable functions will eventually be discovered.

5 Results

We take all unary functions as the starting point. Thanks to earlier investigation in [Dan77] we know the final context. When we investigate all possible implicatively closed partitions such that the implicatively closed subcontext contains all unary functions we find the following:

- We start with 27 unary functions, 26 of them are p-indecomposable;
- After adding all essential counter-examples we obtain 147 functions;
- After using Proposition 6 we obtain 155 functions;
- There remain 42 functions to be discovered. By direct check we find that there does not exist an implicatively closed subcontext containing 155 mentioned above functions such that all the undiscovered functions are second-order irreducible.

Hence, if we start from all unary functions on A_3 all the functions F_3^p will eventually be discovered.

The experiment was conducted three times starting from different initial contexts, all three times the exploration was successful. The exploration stating from a single constant function $f_{3,0}^u$ took 207 steps.

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