Using Linguistic Hedges in L-rough Concept Analysis

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Abstract. We enrich concept-forming operators in **L**-rough Concept Analysis with linguistic hedges which model semantics of logical connectives 'very' and 'slightly'. Using hedges as parameters for the concept-forming operators we are allowed to modify our uncertainty when forming concepts. As a consequence, by selection of these hedges we can control the size of concept lattice.

Keywords: Formal concept analysis; concept lattice; fuzzy set; linguistic hedge; rough set; uncertainty.

1 Introduction

In [2] we presented a framework which allows us to work with positive and negative attributes in the fuzzy setting by applying two unipolar scales for intents – a positive one and a negative one. The positive scale is implicitly modeled by an antitone Galois connection while the negative scale is modeled by an isotone Galois connection. In this paper we extend this approach in two ways.

First, we work with uncertain information. To do this we extend formal fuzzy contexts to contain two truth-degrees for each object-attribute pair. The two truth-degrees represent necessity and possibility of the fact that an object has an attribute. The interval between these degrees represents the uncertainty presented in a given data.

Second, we parametrize the concept-forming operators used in the framework by unary operators called truth-stressing and truth-depressing linguistic hedges. Their intended use is to model semantics of statements *'it is very sure that this attribute belongs to a fuzzy set (intent)'* and *'it is slightly possible that an attribute belongs a fuzzy set (intent)'*, respectively. In the paper, we demonstrate how the hedges influence the size of concept lattice.

2 Preliminaries

In this section we summarize the basic notions used in the paper.

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Residuated Lattices and Fuzzy Sets

We use complete residuated lattices as basic structures of truth-degrees. A complete residuated lattice [4,12,17] is a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, i.e. a partially ordered set in which arbitrary infima and suprema exist; $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is a binary operation which is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$; $\otimes$ and $\rightarrow$ satisfy adjointness, i.e. $a \otimes b \leqslant c$ iff $a \leqslant b \rightarrow c$. 0 and 1 denote the least and greatest elements. The partial order of $\mathbf{L}$ is denoted by $\leqslant$. Throughout this work, $\mathbf{L}$ denotes an arbitrary complete residuated lattice.

Elements of L are called truth degrees. Operations $\otimes$ (multiplication) and $\rightarrow$ (residuum) play the role of (truth functions of) "fuzzy conjunction" and "fuzzy implication". Furthermore, we define the complement of $a \in L$ as $\neg a = a \rightarrow 0$.

An $\mathbf{L}$-set (or fuzzy set) A in a universe set X is a mapping assigning to each $x \in X$ some truth degree $A(x) \in L$. The set of all $\mathbf{L}$-sets in a universe X is denoted $\mathbf{L}^X$.

The operations with $\mathbf{L}$-sets are defined componentwise. For instance, the intersection of $\mathbf{L}$-sets $A, B \in \mathbf{L}^X$ is an $\mathbf{L}$-set $A \cap B$ in X such that $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in X$. An $\mathbf{L}$-set $A \in \mathbf{L}^X$ is also denoted $\{{}^{A(x)}/x \mid x \in X\}$. If for all $y \in X$ distinct from $x_1, \ldots, x_n$ we have $A(y) = 0$, we also write $\{{}^{A(x_1)}/x_1, \ldots, {}^{A(x_n)}/x_n\}$.

An $\mathbf{L}$-set $A \in \mathbf{L}^X$ is called normal if there is $x \in X$ such that $A(x) = 1$. An $\mathbf{L}$-set $A \in \mathbf{L}^X$ is called crisp if $A(x) \in \{0,1\}$ for each $x \in X$. Crisp $\mathbf{L}$-sets can be identified with ordinary sets. For a crisp A, we also write $x \in A$ for $A(x) = 1$ and $x \notin A$ for $A(x) = 0$.

For $A, B \in \mathbf{L}^X$ we define the degree of inclusion of A in B by $S(A,B) = \bigwedge_{x \in X} A(x) \rightarrow B(x)$. Graded inclusion generalizes the classical inclusion relation. Described verbally, $S(A,B)$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A,B) = 1$. As a consequence, we have $A \subseteq B$ iff $A(x) \leqslant B(x)$ for each $x \in X$.

By $\mathbf{L}^{-1}$ we denote L with dual lattice order. An $\mathbf{L}$-*rough set* A in a universe X is a pair of $\mathbf{L}$-sets $A = \langle \underline{A}, \overline{A}\rangle \in (\mathbf{L} \times \mathbf{L}^{-1})^U$. The $\underline{A}$ is called an *lower approximation* of A and the $\overline{A}$ is called a *upper approximation* of A.[1]

The operations with $\mathbf{L}$-rough sets are again defined componentwise, i.e.

$$\bigcap_{i \in I} \langle \underline{A}, \overline{A}\rangle = \langle \bigcap_{i \in I} \underline{A}, \bigcap_{i \in I}^{-1} \overline{A}\rangle = \langle \bigcap_{i \in I} \underline{A}, \bigcup_{i \in I} \overline{A}\rangle,$$
$$\bigcup_{i \in I} \langle \underline{A}, \overline{A}\rangle = \langle \bigcup_{i \in I} \underline{A}, \bigcup_{i \in I}^{-1} \overline{A}\rangle = \langle \bigcup_{i \in I} \underline{A}, \bigcap_{i \in I} \overline{A}\rangle.$$

Similarly, the graded subsethood is then applied componentwise

$$S(\langle \underline{A}, \overline{A}\rangle, \langle \underline{B}, \overline{B}\rangle) = S(\underline{A}, \underline{B}) \wedge S^{-1}(\overline{A}, \overline{B}) = S(\underline{A}, \underline{B}) \wedge S(\overline{B}, \overline{A})$$

[1] In our setting we consider intents to be $\mathbf{L}$-rough sets; the lower and upper approximation are interpreted as necessary intent and possible intent, respectively.

and the crisp subsethood is then defined using the graded subsethood:

$$\langle \underline{A}, \overline{A} \rangle \subseteq \langle \underline{B}, \overline{B} \rangle \text{ iff } S(\langle \underline{A}, \overline{A} \rangle, \langle \underline{B}, \overline{B} \rangle) = 1, \text{ iff } \underline{A} \subseteq \underline{B} \text{ and } \overline{B} \subseteq \overline{A}.$$

An **L**-rough set $\langle \underline{A}, \overline{A} \rangle$ is called *natural* if $\underline{A} \subseteq \overline{A}$.

Binary **L**-relations (binary fuzzy relations) between X and Y can be thought of as **L**-sets in the universe $X \times Y$. That is, a binary **L**-relation $I \in \mathbf{L}^{X \times Y}$ between a set X and a set Y is a mapping assigning to each $x \in X$ and each $y \in Y$ a truth degree $I(x, y) \in L$ (a degree to which x and y are related by I). **L**-rough relations are then $(\mathbf{L} \times \mathbf{L}^{-1})$-sets in $X \times Y$. For **L**-relation $I \in \mathbf{L}^{X \times Y}$ we define its inverse $I^{-1} \in \mathbf{L}^{Y \times X}$ as $I^{-1}(y, x) = I(x, y)$ for all $x \in X$, $y \in Y$.

Formal Concept Analysis in the Fuzzy Setting

An **L**-context is a triplet $\langle X, Y, I \rangle$ where X and Y are (ordinary) sets and $I \in \mathbf{L}^{X \times Y}$ is an **L**-relation between X and Y. Elements of X are called objects, elements of Y are called attributes, I is called an incidence relation. $I(x, y) = a$ is read: "The object x has the attribute y to degree a."

Consider the following pairs of operators induced by an **L**-context $\langle X, Y, I \rangle$. First, the pair $\langle \uparrow, \downarrow \rangle$ of operators $^{\uparrow} : \mathbf{L}^X \to \mathbf{L}^Y$ and $^{\downarrow} : \mathbf{L}^Y \to \mathbf{L}^X$ is defined by

$$A^{\uparrow}(y) = \bigwedge_{x \in X} A(x) \to I(x, y) \quad \text{and} \quad B^{\downarrow}(x) = \bigwedge_{y \in Y} B(y) \to I(x, y).$$

Second, the pair $\langle \cap, \cup \rangle$ of operators $^{\cap} : \mathbf{L}^X \to \mathbf{L}^Y$ and $^{\cup} : \mathbf{L}^Y \to \mathbf{L}^X$ is defined by

$$A^{\cap}(y) = \bigvee_{x \in X} A(x) \otimes I(x, y) \quad \text{and} \quad B^{\cup}(x) = \bigwedge_{y \in Y} I(x, y) \to B(y).$$

To emphasize that the operators are induced by I, we also denote the operators by $\langle \uparrow_I, \downarrow_I \rangle$ and $\langle \cap_I, \cup_I \rangle$.

Fixpoints of these operators are called formal concepts. The set of all formal concepts (along with set inclusion) forms a complete lattice, called **L**-concept lattice. We denote the sets of all concepts (as well as the corresponding **L**-concept lattice) by $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$ and $\mathcal{B}^{\cap\cup}(X, Y, I)$, i.e.

$$\mathcal{B}^{\uparrow\downarrow}(X, Y, I) = \{\langle A, B \rangle \in \mathbf{L}^X \times \mathbf{L}^Y \mid A^{\uparrow} = B,\ B^{\downarrow} = A\},$$
$$\mathcal{B}^{\cap\cup}(X, Y, I) = \{\langle A, B \rangle \in \mathbf{L}^X \times \mathbf{L}^Y \mid A^{\cap} = B,\ B^{\cup} = A\}.$$

For an **L**-concept lattice $\mathcal{B}(X, Y, I)$, where $\mathcal{B}$ is either $\mathcal{B}^{\uparrow\downarrow}$ or $\mathcal{B}^{\cap\cup}$, denote the corresponding sets of extents and intents by $\mathrm{Ext}(X, Y, I)$ and $\mathrm{Int}(X, Y, I)$. That is,

$$\mathrm{Ext}(X, Y, I) = \{A \in \mathbf{L}^X \mid \langle A, B \rangle \in \mathcal{B}(X, Y, I) \text{ for some } B\},$$
$$\mathrm{Int}(X, Y, I) = \{B \in \mathbf{L}^Y \mid \langle A, B \rangle \in \mathcal{B}(X, Y, I) \text{ for some } A\}.$$

An $(\mathbf{L}_1, \mathbf{L}_2)$-Galois connection between the sets X and Y is a pair $\langle f, g \rangle$ of mappings $f : \mathbf{L}_1^X \to \mathbf{L}_2^Y$, $g : \mathbf{L}_2^Y \to \mathbf{L}_1^X$, satisfying

$$S(A, g(B)) = S(B, f(A))$$

for every $A \in \mathbf{L}_1^X, B \in \mathbf{L}_2^Y$.

One can easily observe that the couple $\langle \uparrow, \downarrow \rangle$ forms an $(\mathbf{L}, \mathbf{L})$-Galois connection between X and Y, while $\langle \cap, \cup \rangle$ forms an $(\mathbf{L}, \mathbf{L}^{-1})$-Galois connection between X and Y.

$\mathbf{L}$-rough Contexts and $\mathbf{L}$-rough Concepts Lattices

An $\mathbf{L}$-rough context is a quadruple $\langle X, Y, \underline{I}, \overline{I} \rangle$, where X and Y are (crisp) sets of objects and attributes, respectively, and the $\langle \underline{I}, \overline{I} \rangle$ is a $\mathbf{L}$-rough relation. The meaning of $\langle \underline{I}, \overline{I} \rangle$ is as follows: $\underline{I}(x, y)$ (resp. $\overline{I}(x, y)$) is the truth degree to which the object x surely (resp. possibly) has the attribute y. The quadruple $\langle X, Y, \underline{I}, \overline{I} \rangle$ is called a $\mathbf{L}$-*rough context*.

The $\mathbf{L}$-rough context induces two operators defined as follows. Let $\langle X, Y, \underline{I}, \overline{I} \rangle$ be an $\mathbf{L}$-rough context. Define $\mathbf{L}$-*rough concept-forming operators* as

$$\begin{aligned} A^{\vartriangle} &= \langle A^{\uparrow_{\underline{I}}}, A^{\cap_{\overline{I}}} \rangle, \\ \langle \underline{B}, \overline{B} \rangle^{\triangledown} &= \underline{B}^{\downarrow_{\underline{I}}} \cap \overline{B}^{\cup_{\overline{I}}} \end{aligned} \tag{1}$$

for $A \in \mathbf{L}^X, \underline{B}, \overline{B} \in \mathbf{L}^Y$. Fixed points of $\langle \vartriangle, \triangledown \rangle$, i.e. tuples $\langle A, \langle \underline{B}, \overline{B} \rangle \rangle \in \mathbf{L}^X \times (\mathbf{L} \times \mathbf{L}^{-1})^Y$ such that $A^{\vartriangle} = \langle \underline{B}, \overline{B} \rangle$ and $\langle \underline{B}, \overline{B} \rangle^{\triangledown} = A$, are called $\mathbf{L}$-*rough concepts*. The $\underline{B}$ and $\overline{B}$ are called *lower intent approximation* and *upper intent approximation*, respectively.

In [2] we showed that the pair of operators (1) is an $(\mathbf{L}, \mathbf{L} \times \mathbf{L}^{-1})$-Galois connection.

Linguistic Hedges

Truth-stressing hedges were studied from the point of fuzzy logic as logical connectives 'very true', see [13]. Our approach is close to that in [13]. A *truth-stressing hedge* is a mapping ${}^* : L \to L$ satisfying

$$1^* = 1, \quad a^* \leqslant a, \quad a \leqslant b \text{ implies } a^* \leqslant b^*, \quad a^{**} = a^* \tag{2}$$

for each $a, b \in L$. Truth-stressing hedges were used to parametrize antitone $\mathbf{L}$-Galois connections e.g. in [3, 5, 9], and also to parameterize isotone $\mathbf{L}$-Galois connections in [1].

On every complete residuated lattice $\mathbf{L}$, there are two important truth-stressing hedges:

(i) identity, i.e. $a^* = a$ $(a \in L)$;
(ii) globalization, i.e.

$$a^* = \begin{cases} 1, & \text{if } a = 1, \\ 0, & \text{otherwise.} \end{cases}$$

A *truth-depressing hedge* is a mapping ${}^\square : L \to L$ such that following conditions are satisfied

$$0^\square = 0, \quad a \leqslant a^\square, \quad a \leqslant b \text{ implies } a^\square \leqslant b^\square, \quad a^{\square\square} = a^\square$$

for each $a, b \in L$. A truth-depressing hedge is a (truth function of) logical connective 'slightly true', see [16].

On every complete residuated lattice $\mathbf{L}$, there are two important truth-depressing hedges:

(i) identity, i.e. $a^{\square} = a$ $(a \in L)$;
(ii) antiglobalization, i.e.

$$a^{\square} = \begin{cases} 0, & \text{if } a = 0, \\ 1, & \text{otherwise} . \end{cases}$$

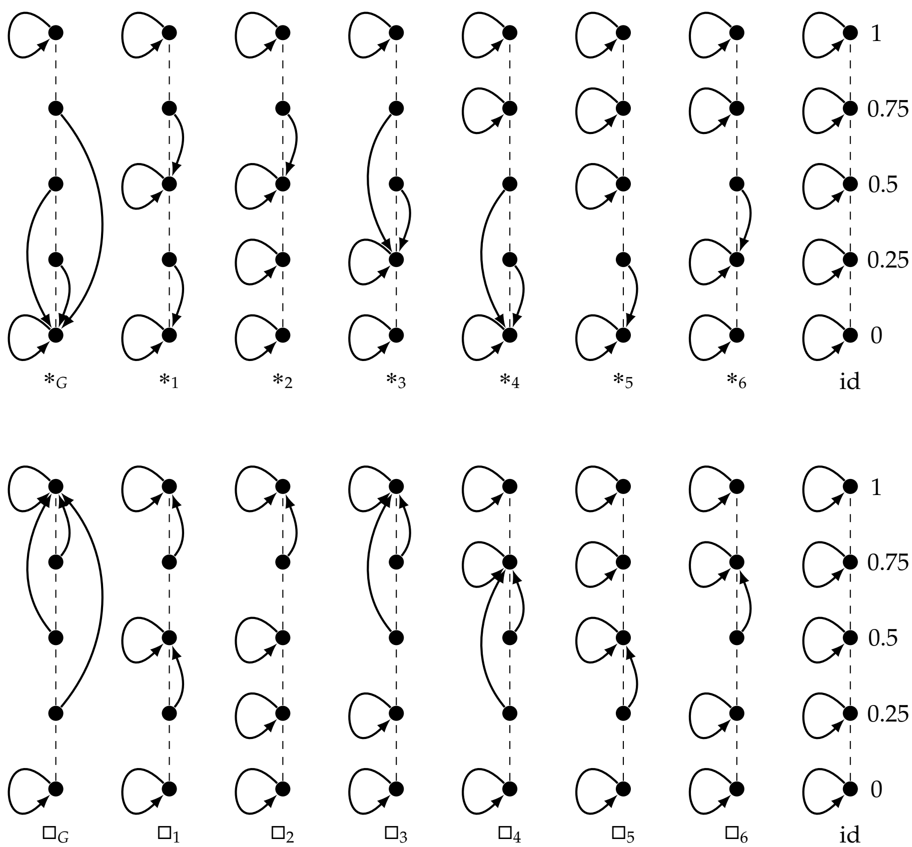

Fig. 1. Truth-stressing hedges (top) and truth-depressing hedges (bottom) on 5-element chain with Łukasiewicz operations $\mathbf{L} = \langle\{0, 0.25, 0.5, 0.75, 1\}, \min, \max, \otimes, \rightarrow, 0, 1\rangle$. The leftmost truth-stressing hedge $*_G$ is the globalization, leftmost truth-depressing hedge $\square_G$ is the antiglobalization. The rightmost hedges denoted by id are the identities.

For truth-stressing/truth-depressing hedge $*$ we denote by $\mathrm{fix}(*)$ set of its idempotent elements in $\mathbf{L}$; i.e. $\mathrm{fix}(*) = \{a \in L \mid a^* = a\}$.

Let $*_1, *_2$ be truth-stressing hedges on $\mathbf{L}$ such that $\mathrm{fix}(*_1) \subseteq \mathrm{fix}(*_2)$; then for each $a \in A$, $a^{*_1 *_2} = a^{*_1}$ holds. The same holds true for $*_1, *_2$ being truth-depressing hedges.

We naturally extend application of truth-stressing/truth-depressing hedges to $\mathbf{L}$-sets: $A^*(x) = A(x)^*$ for all $x \in U$.

3 Results

The $\mathbf{L}$-rough concept-forming operator $\vartriangle$ gives for each $\mathbf{L}$-set of objects two $\mathbf{L}$-sets of attributes. The first one represents a necessity of having the attributes and second one a possibility of having the attributes. We add linguistic hedges to the concept-forming operators to control shape of the two $\mathbf{L}$-sets.

Since the $\mathbf{L}$-rough concept-forming operators are defined via $\langle\uparrow,\downarrow\rangle$ and $\langle\cap,\cup\rangle$, we first recall the parametrization of these operators as described in [8, 15].

3.1 Linguistic Hedges in Formal Fuzzy Concept Analysis

Let $\langle X, Y, I\rangle$ be an $\mathbf{L}$-context and let $\heartsuit, \diamondsuit$ be truth-stressing hedges on $\mathbf{L}$. The antitone concept-forming operators parametrized by $\heartsuit$ and $\diamondsuit$ induced by I are defined as

$$A^{\uparrow_\heartsuit}(y) = \bigwedge_{x \in X} A(x)^\heartsuit \to I(x, y),$$
$$B^{\downarrow_\diamondsuit}(x) = \bigwedge_{y \in Y} B(y)^\diamondsuit \to I(x, y)$$

for all $A \in \mathbf{L}^X, B \in \mathbf{L}^Y$.

Let $\heartsuit$ and $\spadesuit$ be truth-stressing hedge and truth-depressing hedge on $\mathbf{L}$, respectively. The isotone concept-forming operators parametrized by $\heartsuit$ and $\spadesuit$ induced by I are defined as

$$A^{\cap_\heartsuit}(y) = \bigvee_{x \in X} A(x)^\heartsuit \otimes I(x, y),$$
$$B^{\cup_\spadesuit}(x) = \bigwedge_{y \in Y} I(x, y) \to B(y)^\spadesuit$$

for all $A \in \mathbf{L}^X, B \in \mathbf{L}^Y$.

Properties of the hedges in the setting of multi-adjoint concept lattices with heterogeneous conjunctors were studied in [14].

3.2 L-rough Concept-Forming Operators with Linguistic Hedges

Let $\heartsuit, \diamondsuit$ be truth-stressing hedges on $\mathbf{L}$ and let $\spadesuit$ be a truth-depressing hedge on $\mathbf{L}$. We parametrize the $\mathbf{L}$-rough concept-forming operators as

$$A^\blacktriangle = \langle A^{\uparrow_\heartsuit}, A^{\cap_\heartsuit}\rangle \quad \text{and} \quad \langle \underline{B}, \overline{B}\rangle^\blacktriangledown = \underline{B}^{\downarrow_\diamondsuit} \cap \overline{B}^{\cup_\spadesuit} \qquad (3)$$

for $A \in \mathbf{L}^X, \underline{B}, \overline{B} \in \mathbf{L}^Y$.

Remark 1. When the all three hedges are identities the pair $\langle \blacktriangle, \blacktriangledown \rangle$ is equivalent to $\langle \vartriangle, \triangledown \rangle$; so it is an $(\mathbf{L}, \mathbf{L} \times \mathbf{L}^{-1})$-Galois connection. For arbitrary hedges this does not hold.

The following theorem describes properties of $\langle \blacktriangle, \blacktriangledown \rangle$.

Theorem 1. *The pair $\langle \blacktriangle, \blacktriangledown \rangle$ of $\mathbf{L}$-rough concept-forming operators parametrized by hedges has the following properties.*

(a) $A^{\blacktriangle} = A^{\heartsuit\vartriangle} = A^{\heartsuit\blacktriangle}$ *and* $\langle \underline{B}, \overline{B} \rangle^{\blacktriangledown} = \langle \underline{B}^{\blacklozenge}, \overline{B}^{\spadesuit} \rangle^{\triangledown} = \langle \underline{B}^{\blacklozenge}, \overline{B}^{\spadesuit} \rangle^{\blacktriangledown}$
(b) $A^{\vartriangle} \subseteq A^{\blacktriangle}$ *and* $\langle \underline{B}, \overline{B} \rangle^{\triangledown} \subseteq \langle \underline{B}, \overline{B} \rangle^{\blacktriangledown}$
(c) $S(A_1^{\heartsuit}, A_2^{\heartsuit}) \leqslant S(A_2^{\blacktriangle}, A_1^{\blacktriangle})$ *and* $S(\langle \underline{B_1}, \overline{B_1} \rangle, \langle \underline{B_2}, \overline{B_2} \rangle) \leqslant S(\langle \underline{B_2}, \overline{B_2} \rangle^{\blacktriangledown}, \langle \underline{B_1}, \overline{B_1} \rangle^{\blacktriangledown})$
(d) $A^{\heartsuit} \subseteq A^{\blacktriangle\blacktriangledown}$ *and* $\langle \underline{B}^{\blacklozenge}, \overline{B}^{\spadesuit} \rangle \subseteq \langle \underline{B}, \overline{B} \rangle^{\blacktriangledown\blacktriangle}$;
(e) $A_1 \subseteq A_2$ *implies* $A_2^{\blacktriangle} \subseteq A_1^{\blacktriangle}$ *and* $\langle \underline{B_1}, \overline{B_1} \rangle \subseteq \langle \underline{B_2}, \overline{B_2} \rangle$ *implies* $\langle \underline{B_2}, \overline{B_2} \rangle^{\blacktriangledown} \subseteq \langle \underline{B_1}, \overline{B_1} \rangle^{\blacktriangledown}$
(f) $S(A^{\heartsuit}, \langle \underline{B}, \overline{B} \rangle^{\blacktriangledown}) = S(\langle \underline{B}^{\blacklozenge}, \overline{B}^{\spadesuit} \rangle, A^{\blacktriangle})$
(g) $(\bigcup_{i \in I} A_i^{\heartsuit})^{\blacktriangle} = \bigcap_{i \in I} A_i^{\blacktriangle}$ *and* $(\langle \bigcup_{i \in I} \underline{B_i}^{\blacklozenge}, \bigcap_{i \in I} \overline{B_i}^{\spadesuit} \rangle)^{\blacktriangledown} = \bigcap_{i \in I} \langle \underline{B_i}, \overline{B_i} \rangle^{\blacktriangledown}$
(h) $A^{\blacktriangle\blacktriangledown} = A^{\blacktriangle\blacktriangledown\blacktriangle\blacktriangledown}$ *and* $\langle \underline{B}, \overline{B} \rangle^{\blacktriangledown\blacktriangle} = \langle \underline{B}, \overline{B} \rangle^{\blacktriangledown\blacktriangle\blacktriangledown\blacktriangle}$.

Proof. (a) Follows immediately from definition of $\blacktriangle$ and $\blacktriangledown$ and idempotency of hedges.

(b) From (2) we have $A^{\heartsuit} \subseteq A$; by properties of Galois connections the inclusion implies $A^{\vartriangle} \subseteq A^{\heartsuit\vartriangle}$, which is by (a) equivalent to $A^{\vartriangle} \subseteq A^{\blacktriangle}$. Proof of the second statement in (b) is similar.

(c) Follows from (a) and properties of Galois connections.

(d) By [2, Corollary 1(a)] we have $A^{\heartsuit} \subseteq A^{\heartsuit\vartriangle\triangledown}$. Using (a) we get $A^{\heartsuit} \subseteq A^{\blacktriangle\triangledown}$ and from (b) we have $A^{\blacktriangle\triangledown} \subseteq A^{\blacktriangle\blacktriangledown}$, so $A^{\heartsuit} \subseteq A^{\blacktriangle\blacktriangledown}$. Similarly for the second claim.

(e) Follows directly from [2, Corollary 1(c)] and properties of Galois connections.

(f) Since $\langle \vartriangle, \triangledown \rangle$ forms $(\mathbf{L}, \mathbf{L} \times \mathbf{L}^{-1})$-Galois connection and using (a) we have $S(A^{\heartsuit}, \langle \underline{B}, \overline{B} \rangle^{\blacktriangledown}) = S(A^{\heartsuit}, \langle \underline{B}^{\blacklozenge}, \overline{B}^{\spadesuit} \rangle^{\triangledown}) = S(\langle \underline{B}^{\blacklozenge}, \overline{B}^{\spadesuit} \rangle, A^{\heartsuit\vartriangle}) = S(\langle \underline{B}^{\blacklozenge}, \overline{B}^{\spadesuit} \rangle, A^{\blacktriangle})$.

(g) We can easily get

$$\begin{aligned}(\bigcup_{i \in I} A_i^{\heartsuit})^{\blacktriangle} &= \langle (\bigcup_{i \in I} A_i^{\heartsuit})^{\uparrow\heartsuit}, (\bigcup_{i \in I} A_i^{\heartsuit})^{\cap\heartsuit} \rangle = \langle \bigcap_{i \in I} A_i^{\uparrow\heartsuit}, \bigcup_{i \in I} A_i^{\cap\heartsuit} \rangle \\ &= \bigcap_{i \in I} \langle A_i^{\uparrow\heartsuit}, A_i^{\cap\heartsuit} \rangle = \bigcap_{i \in I} A_i^{\blacktriangle},\end{aligned}$$

and

$$\begin{aligned}(\langle \bigcup_{i \in I} \underline{B_i}^{\blacklozenge}, \bigcap_{i \in I} \overline{B_i}^{\spadesuit} \rangle)^{\blacktriangledown} &= (\bigcup_{i \in I} \underline{B_i}^{\blacklozenge})^{\downarrow\blacklozenge} \cap (\bigcap_{i \in I} \overline{B_i}^{\spadesuit})^{\cup\spadesuit} = \bigcap_{i \in I} \underline{B_i}^{\downarrow\blacklozenge} \cap \bigcap_{i \in I} \overline{B_i}^{\cup\spadesuit} \\ &= \bigcap_{i \in I} (\underline{B_i}^{\downarrow\blacklozenge} \cap \overline{B_i}^{\cup\spadesuit}) = \bigcap_{i \in I} \langle \underline{B_i}, \overline{B_i} \rangle^{\blacktriangledown}.\end{aligned}$$

(h) Using (a), (d) and (e) twice, we have $A^{\blacktriangle\blacktriangledown} \subseteq A^{\blacktriangle\blacktriangledown\blacktriangle\blacktriangledown}$. Using (d) for $\langle \underline{B}, \overline{B} \rangle = A^{\blacktriangle}$ we have $A^{\blacktriangle\heartsuit} \subseteq A^{\blacktriangle\heartsuit\blacktriangledown\blacktriangle} = A^{\blacktriangle\blacktriangledown\blacktriangle}$. Then applying (e) we get $A^{\blacktriangle\blacktriangledown\blacktriangle\blacktriangledown} \subseteq A^{\blacktriangle\blacktriangledown}$ proving the first claim. The second claim can be proved analogically. □

The set of fixed points of $\langle \blacktriangle, \blacktriangledown \rangle$ endowed with partial order $\leqslant$ given by

$$\begin{aligned} \langle A_1, \underline{B}_1, \overline{B}_1 \rangle \leqslant \langle A_2, \underline{B}_2, \overline{B}_2 \rangle \quad & \text{iff} \quad A_1 \subseteq A_2 \\ & \text{iff} \quad \langle \underline{B}_1, \overline{B}_1 \rangle \subseteq \langle \underline{B}_2, \overline{B}_2 \rangle \end{aligned} \tag{4}$$

is denoted by $\mathcal{B}^{\blacktriangle\blacktriangledown}_{\heartsuit,\diamondsuit,\spadesuit}(X, Y, \underline{I}, \overline{I})$.

Remark 2. Note that from (4) it is clear that if a concept has non-natural **L**-rough intent then all its subconcepts have non-natural intent. If such concepts are not desired, one can simply ignore them and work with the iceberg lattice of concepts with natural **L**-rough intents.

The next theorem shows a crisp representation of $\mathcal{B}^{\blacktriangle\blacktriangledown}_{\heartsuit,\diamondsuit,\spadesuit}(X, Y, \underline{I}, \overline{I})$.

Theorem 2. *$\mathcal{B}^{\blacktriangle\blacktriangledown}_{\heartsuit,\diamondsuit,\spadesuit}(X, Y, \underline{I}, \overline{I})$ is isomorphic to ordinary concept lattice $\mathcal{B}^{\uparrow\downarrow}(X \times \mathrm{fix}(\heartsuit), Y \times \mathrm{fix}(\diamondsuit) \times \mathrm{fix}(\spadesuit), I^{\times})$ where*

$$\langle \langle x, a \rangle, \langle y, \underline{b}, \overline{b} \rangle \rangle \in I^{\times} \text{ iff } a \otimes \underline{b} \leqslant \underline{I}(x, y) \text{ and } a \to \overline{b} \geqslant \overline{I}(x, y).$$

Proof. This proof can be done by following the same steps as in [8, 15]. □

The following theorem explains the structure of $\mathcal{B}^{\blacktriangle\blacktriangledown}_{\heartsuit,\diamondsuit,\spadesuit}(X, Y, \underline{I}, \overline{I})$.

Theorem 3. *$\mathcal{B}^{\blacktriangle\blacktriangledown}_{\heartsuit,\diamondsuit,\spadesuit}(X, Y, \underline{I}, \overline{I})$ is a complete lattice with suprema and infima defined as*

$$\bigwedge_i \langle A_i, \langle \underline{B_i}, \overline{B_i} \rangle \rangle = \langle (\bigcap_i A_i)^{\blacktriangle\blacktriangledown}, \langle \bigcup_i \underline{B_i}^{\diamondsuit}, \bigcap_i \overline{B_i}^{\spadesuit} \rangle^{\blacktriangledown\blacktriangle} \rangle,$$

$$\bigvee_i \langle A_i, \langle \underline{B_i}, \overline{B_i} \rangle \rangle = \langle (\bigcup_i A_i^{\heartsuit})^{\blacktriangle\blacktriangledown}, \langle \bigcap_i \underline{B_i}, \bigcup_i \overline{B_i} \rangle^{\blacktriangledown\blacktriangle} \rangle$$

for all $A_i \in \mathbf{L}^X, \underline{B}_i \in \mathbf{L}^Y, \overline{B}_i \in \mathbf{L}^Y$.

Proof. Follows from Theorem 2. □

Remark 3. Note that if we alternatively define (3) as

$$A^{\blacktriangle} = \langle (A^{\uparrow\heartsuit})^{\diamondsuit}, (A^{\cap\heartsuit})^{\spadesuit} \rangle \quad \text{and} \quad \langle \underline{B}, \overline{B} \rangle^{\blacktriangledown} = (\underline{B}^{\downarrow\diamondsuit} \cap \overline{B}^{\cup\spadesuit})^{\heartsuit} \tag{5}$$

or

$$A^{\blacktriangle} = \langle (A^{\uparrow})^{\diamondsuit}, (A^{\cap})^{\spadesuit} \rangle \quad \text{and} \quad \langle \underline{B}, \overline{B} \rangle^{\blacktriangledown} = (\underline{B}^{\downarrow} \cap \overline{B}^{\cup})^{\heartsuit} \tag{6}$$

or

$$A^{\blacktriangle} = \langle (A^{\uparrow\heartsuit})^{\diamondsuit}, (A^{\cap\heartsuit})^{\spadesuit} \rangle \quad \text{and} \quad \langle \underline{B}, \overline{B} \rangle^{\blacktriangledown} = \langle \underline{B}, \overline{B} \rangle^{\triangledown}$$

or

$$A^{\blacktriangle} = A^{\vartriangle} \quad \text{and} \quad \langle \underline{B}, \overline{B} \rangle^{\blacktriangledown} = (\underline{B}^{\downarrow\diamondsuit} \cap \overline{B}^{\cup\spadesuit})^{\heartsuit}$$

we obtain an isomorphic concept lattice. In addition (5) and (6) produce the same concept lattice.

3.3 Size Reduction of Fuzzy Rough Concept Lattices

This part provides analogous results on reduction with truth-stressing and truth-depressing hedges as [10] for antitone fuzzy concept-forming operators and [15] for isotone fuzzy concept-forming operators.

For the next theorem we need the following lemma.

Lemma 1. *Let* $\varheartsuit, \heartsuit, \vardiamondsuit, \diamondsuit$ *be truth-stressing hedges on* $\mathbf{L}$ *such that* $\mathrm{fix}(\varheartsuit) \subseteq \mathrm{fix}(\heartsuit), \mathrm{fix}(\vardiamondsuit) \subseteq \mathrm{fix}(\diamondsuit)$*; let* $\spadesuit, \varspadesuit$ *be truth-depressing hedges on* $\mathbf{L}$ *such that and* $\mathrm{fix}(\spadesuit) \subseteq \mathrm{fix}(\varspadesuit)$*. We have*

$$A^{\blacktriangle_{\heartsuit}} \subseteq A^{\blacktriangle_{\varheartsuit}} \quad \textit{and} \quad \langle \underline{B}, \overline{B} \rangle^{\blacktriangledown_{\diamondsuit,\varspadesuit}} \subseteq \langle \underline{B}, \overline{B} \rangle^{\blacktriangledown_{\vardiamondsuit,\spadesuit}}.$$

Proof. We have $A^{\varheartsuit\heartsuit} \subseteq A^{\heartsuit}$ from (2). From the assumption $\mathrm{fix}(\varheartsuit) \subseteq \mathrm{fix}(\heartsuit)$ we get $A^{\varheartsuit\heartsuit} = A^{\varheartsuit}$; whence we have $A^{\varheartsuit} \subseteq A^{\heartsuit}$. Theorem 1(e) implies $A^{\heartsuit\blacktriangle} \subseteq A^{\varheartsuit\blacktriangle}$ which is by the claim (a) of this theorem equivalent to $A^{\blacktriangle_{\heartsuit}} \subseteq A^{\blacktriangle_{\varheartsuit}}$. The second claim can be proved similarly. □

Theorem 4. *Let* $\varheartsuit, \heartsuit, \vardiamondsuit, \diamondsuit$ *be truth-stressing hedges on* $\mathbf{L}$ *such that* $\mathrm{fix}(\varheartsuit) \subseteq \mathrm{fix}(\heartsuit)$, $\mathrm{fix}(\vardiamondsuit) \subseteq \mathrm{fix}(\diamondsuit)$*; let* $\spadesuit, \varspadesuit$ *be truth-depressing hedges on* $\mathbf{L}$ *s.t. and* $\mathrm{fix}(\spadesuit) \subseteq \mathrm{fix}(\varspadesuit)$,

$$|\mathcal{B}^{\blacktriangle\blacktriangledown}_{\varheartsuit,\vardiamondsuit,\spadesuit}(X, Y, \underline{I}, \overline{I})| \leqslant |\mathcal{B}^{\blacktriangle\blacktriangledown}_{\heartsuit,\diamondsuit,\varspadesuit}(X, Y, \underline{I}, \overline{I})|$$

for all $\mathbf{L}$*-rough contexts* $\langle X, Y, \underline{I}, \overline{I} \rangle$.

In addition, if $\varheartsuit = \heartsuit = \mathrm{id}$*, we have*

$$\mathrm{Ext}^{\blacktriangle\blacktriangledown}_{\varheartsuit,\vardiamondsuit,\spadesuit}(X, Y, \underline{I}, \overline{I}) \subseteq \mathrm{Ext}^{\blacktriangle\blacktriangledown}_{\heartsuit,\diamondsuit,\varspadesuit}(X, Y, \underline{I}, \overline{I}).$$

Similarly, if $\vardiamondsuit = \diamondsuit = \spadesuit = \varspadesuit = \mathrm{id}$*, we have*

$$\mathrm{Int}^{\blacktriangle\blacktriangledown}_{\varheartsuit,\vardiamondsuit,\spadesuit}(X, Y, \underline{I}, \overline{I}) \subseteq \mathrm{Int}^{\blacktriangle\blacktriangledown}_{\heartsuit,\diamondsuit,\varspadesuit}(X, Y, \underline{I}, \overline{I}).$$

Proof. (4) follows directly from Theorem 2 and results on subcontexts in [11].

Now, we show (4). Note that each $A \in \mathrm{Ext}^{\blacktriangle\blacktriangledown}_{\varheartsuit,\vardiamondsuit,\spadesuit}(X, Y, \underline{I}, \overline{I})$ we have

$$A = A^{\blacktriangle_{\varheartsuit}\blacktriangledown_{\vardiamondsuit,\spadesuit}} = A^{\blacktriangle_{\heartsuit}\blacktriangledown_{\vardiamondsuit,\spadesuit}} \supseteq A^{\blacktriangle_{\heartsuit}\blacktriangledown_{\diamondsuit,\varspadesuit}} \supseteq A.$$

Thus we have $A \in \mathrm{Ext}^{\blacktriangle\blacktriangledown}_{\heartsuit,\diamondsuit,\varspadesuit}(X, Y, \underline{I}, \overline{I})$. The inclusion (4) can be proved similarly. □

Example 1. Consider the truth-stressing hedges $*_G, *_1, *_2$, id and truth-depressing hedges $\square_G, \square_1, \square_2$, id from Figure 1. One can easily observe that

$$\mathrm{fix}(*_G) \subseteq \mathrm{fix}(*_1) \subseteq \mathrm{fix}(*_2) \subseteq \mathrm{fix}(\mathrm{id})$$
$$\mathrm{fix}(\square_G) \subseteq \mathrm{fix}(\square_1) \subseteq \mathrm{fix}(\square_2) \subseteq \mathrm{fix}(\mathrm{id}).$$

Consider the $\mathbf{L}$-context of books and their graded properties in Fig. 2 with $\mathbf{L}$ being 5-element Łukasiewicz chain. Using various combinations of the hedges we obtain a smooth transition in size of the associated fuzzy rough concept lattice going from 10 concepts up to 498 (see Tab. 1). When the 5-element Gödel chain is used instead, we again get a transition going from 10 concepts up to 298 (see Tab. 2).

	High rating	Large no. of pages	Low price	Top sales rank
1	0.75	0	1	0
2	0.5	1	0.25	0.5
3	1	1	0.25	0.5
4	0.75	0.5	0.25	1
5	0.75	0.25	0.75	0
6	1	0	0.75	0.25

Fig. 2. **L**-context of books and their graded properties; this **L**-context was used in [1, 15] to demonstrate reduction of **L**-concept lattices using hedges.

$\spadesuit = \square_G$	$*_G$	$*_1$	$*_2$	id
$*_G$	10	16	59	61
$*_1$	12	22	65	93
$*_2$	15	26	69	103
id	19	41	97	152

$\spadesuit = \square_1$	$*_G$	$*_1$	$*_2$	id
$*_G$	15	28	71	110
$*_1$	15	28	71	170
$*_2$	22	28	79	195
id	28	28	110	264

$\spadesuit = \square_2$	$*_G$	$*_1$	$*_2$	id
$*_G$	15	53	134	211
$*_1$	15	53	134	290
$*_2$	22	63	146	327
id	28	80	181	415

$\spadesuit = \mathrm{id}$	$*_G$	$*_1$	$*_2$	id
$*_G$	27	75	160	297
$*_1$	27	75	160	372
$*_2$	32	80	165	396
id	40	99	202	498

Table 1. Numbers of concepts in **L**-context from Fig. 2 formed by $\langle \blacktriangle, \blacktriangledown \rangle$ parametrized by ♥, ♦, and ♠. A 5-element Łukasiewicz chain is used as the structure of truth degrees. The rows represent the hedge ♥ and the columns represent the hedge ♦.

$\spadesuit = \square_G$	$*_G$	$*_1$	$*_2$	id
$*_G$	10	18	24	24
$*_1$	12	21	33	36
$*_2$	15	29	45	48
id	19	33	51	54

$\spadesuit = \square_G$	$*_G$	$*_1$	$*_2$	id
$*_G$	15	29	36	45
$*_1$	15	32	49	63
$*_2$	22	57	78	106
id	28	66	89	117

$\spadesuit = \square_G$	$*_G$	$*_1$	$*_2$	id
$*_G$	15	32	48	59
$*_1$	15	32	59	75
$*_2$	22	57	88	118
id	28	66	100	130

$\spadesuit = \square_G$	$*_G$	$*_1$	$*_2$	id
$*_G$	27	50	66	125
$*_1$	27	50	80	167
$*_2$	32	79	113	257
id	40	90	127	298

Table 2. Numbers of concepts in **L**-context from Fig. 2 formed by $\langle \blacktriangle, \blacktriangledown \rangle$ parametrized by ♥, ♦, and ♠. A 5-element Gödel chain is used as the structure of truth degrees. The rows represent the hedge ♥ and the columns represent the hedge ♦.

4 Conclusion and further research

We have shown that the **L**-rough concept-forming operators can be parameterized by truth-stressing and truth-depressing hedges similarly as the antitone and isotone fuzzy concept-forming operators.

Our future research includes a study of attribute implications using whose semantics is related to the present setting. That will combine results on fuzzy attribute implications [7] and attribute containment formulas [6].

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