Pattern Structures and Their Morphisms

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Abstract. Projections of pattern structures don't always lead to pattern structures, however residual projections and o-projections do. As a unifying approach, we introduce the notion of pattern morphisms between pattern structures and provide a general sufficient condition for a homomorphic image of a pattern structure being again a pattern structure. In particular, we receive a better understanding of the theory of o-projections.

1 Introduction

Pattern structures within the framework of formal concept analysis have been introduced in [3]. Since then they have turned out to be a useful tool for analysing various real-world applications (cf. [3–7]). In this work we want to point out that the theoretical foundations of pattern structures encourage still some fruitful discussions. In particular, the role projections play within pattern structures for information reduction still needs some further investigation.

The goal of our paper is to establish an adequate concept of pattern morphism between pattern structures, which also gives a better understanding of the concept of o-projections as recently introduced and investigated in [2]. In [8], we showed that projections of pattern structures do not necessarily lead to pattern structures again, however, residual projections do. It turns out that the concept of residual maps between the posets of patterns (w.r.t. two pattern structures) gives the key for a unifying view of o-projections and residual projections.

We also derive that a pattern morphism from a pattern structure to a pattern setup (introduced in this paper), which is surjective on the sets of objects, yields again a pattern structure.

Our main result states that a pattern morphism always induces an adjunction between the corresponding concept lattices. In case the underlying map between the sets of objects is surjective, the induced residuated map between the concept lattices turns out to be surjective too.

The fundamental order theoretic concepts of our paper are nicely presented in the book on *Residuation Theory* by T.S. Blythe and M.F. Janowitz (cf. [1]).

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2 Preliminaries

Definition 1 (Adjunction)**.** *Let $\mathbb{P} = (P, \leqslant)$ and $\mathbb{L} = (L, \leqslant)$ be posets; furthermore let $f : P \to L$ and $g : L \to P$ be maps.*

(1) The pair (f, g) is an **adjunction** *w.r.t. $(\mathbb{P}, \mathbb{L})$ if $fx \leqslant y$ is equivalent to $x \leqslant gy$ for all $x \in P$ and $y \in L$. In this case, we will refer to $(\mathbb{P}, \mathbb{L}, f, g)$ as a* **poset adjunction**.

(2) f is **residuated** *from $\mathbb{P}$ to $\mathbb{L}$ if the preimage of a principal ideal in $\mathbb{L}$ under f is always a principal ideal in $\mathbb{P}$, that is, for every $y \in L$ there exists $x \in P$ s.t.*

$$f^{-1}\{t \in L \mid t \leqslant y\} = \{s \in P \mid s \leqslant x\}.$$

(3) g is **residual** *from $\mathbb{L}$ to $\mathbb{P}$ if the preimage of a principal filter in $\mathbb{P}$ under g is always a principal filter in $\mathbb{L}$, that is, for every $x \in P$ there exists $y \in L$ s.t.*

$$g^{-1}\{s \in P \mid x \leqslant s\} = \{t \in L \mid y \leqslant t\}.$$

(4) The dual of $\mathbb{L}$ is given by $\mathbb{L}^{\mathrm{op}} = (L, \geqslant)$ with $\geqslant := \{(x,t) \in L \times L \mid t \leqslant x\}$. The pair (f, g) is a **Galois connection** *w.r.t. $(\mathbb{P}, \mathbb{L})$ if (f, g) is an* **adjunction** *w.r.t. $(\mathbb{P}, \mathbb{L}^{\mathrm{op}})$.*

The following well-known facts are straightforward (cf. [1]).

Proposition 1. *Let $\mathbb{P} = (P, \leqslant)$ and $\mathbb{L} = (L, \leqslant)$ be posets.*

(1) A map $f : P \to L$ is residuated from $\mathbb{P}$ to $\mathbb{L}$ iff there exists a map $g : L \to P$ s.t. (f, g) is an adjunction w.r.t. $(\mathbb{P}, \mathbb{L})$.

(2) A map $g : L \to P$ is residual from $\mathbb{L}$ to $\mathbb{P}$ iff there exists a map $f : P \to L$ s.t. (f, g) is an adjunction w.r.t. $(\mathbb{P}, \mathbb{L})$.

(3) If (f, g) and (h, k) are adjunctions w.r.t. $(\mathbb{P}, \mathbb{L})$ with $f = h$ or $g = k$ then $f = h$ and $g = k$.

(4) If f is a residuated map from $\mathbb{P}$ to $\mathbb{L}$, then there exists a unique residual map f^+ from $\mathbb{L}$ to $\mathbb{P}$ s.t. (f, f^+) is an adjunction w.r.t. $(\mathbb{P}, \mathbb{L})$. In this case, f^+ is called the **residual map** *of f.*

(5) If g is a residual map from $\mathbb{L}$ to $\mathbb{P}$, then there exists a unique residuated map g^- from $\mathbb{P}$ to $\mathbb{L}$ s.t. (g^-, g) is an adjunction w.r.t. $(\mathbb{P}, \mathbb{L})$. In this case, g^- is called the **residuated map** *of g.*

Definition 2. *Let $\mathbb{P} = (P, \leqslant)$ be a poset and $T \subseteq P$. Then*

(1) The **restriction** *of $\mathbb{P}$ onto T is given by $\mathbb{P}|T := (T, \leqslant \cap (T \times T))$, which clearly is a poset too.*

(2) The **canonical embedding** *of $\mathbb{P}|T$ into $\mathbb{P}$ is given by the map $T \to P, t \mapsto t$.*

(3) T is a **kernel system** *in $\mathbb{P}$ if the canonical embedding τ of $\mathbb{P}|T$ into $\mathbb{P}$ is residuated. In this case, the residual map φ of τ will also be called the* **residual map** *of T in $\mathbb{P}$. The composition $\kappa := \tau \circ \varphi$ is referred to as the* **kernel operator** *associated with T in $\mathbb{P}$.*

(4) Dually, T is a **closure system** *in $\mathbb{P}$ if the cannonical embedding τ of $\mathbb{P}|T$ into $\mathbb{P}$ is residual. In this case, the residuated map ψ of τ will also be called the* **residuated map** *of T in $\mathbb{P}$. The composition $\gamma := \tau \circ \psi$ is referred to as the* **closure operator** *associated with T in $\mathbb{P}$.*

(5) *A map* $\kappa : P \to P$ *is a* **kernel operator** *on* $\mathbb{P}$ *if* $s \leqslant x$ *is equivalent to* $s \leqslant \kappa x$ *for all* $s \in \kappa P$ *and* $x \in P$.
Remark: In this case, κP *forms a kernel system in* $\mathbb{P}$*, the kernel operator of which is* κ.

(6) *Dually, a map* $\gamma : P \to P$ *is a* **closure operator** *on* $\mathbb{P}$ *if* $x \leqslant t$ *is equivalent to* $\gamma x \leqslant t$ *for all* $x \in P$ *and* $t \in \gamma P$.
Remark: In this case, φP *forms a closure system in* $\mathbb{P}$*, the closure operator of which is* γ.

The following known facts will be needed for the sequel (cf. [1]) .

Proposition 2. *Let* $\mathbb{P} = (P, \leqslant)$ *and* $\mathbb{L} = (L, \leqslant)$ *be posets.*

(1) *If* f *is a residuated map from* $\mathbb{P}$ *to* $\mathbb{L}$ *then* f *preserves all existing suprema in* $\mathbb{P}$*, that is, if* $s \in P$ *is the supremum (least upper bound) of* $X \subseteq P$ *in* $\mathbb{P}$ *then* fs *is the supremum of* fX *in* $\mathbb{L}$.
In case $\mathbb{P}$ *and* $\mathbb{L}$ *are complete lattices, the reverse holds too: If a map* f *from* $\mathbb{P}$ *to* $\mathbb{L}$ *preserves all suprema, that is,*

$$f(\sup_{\mathbb{P}} X) = \sup_{\mathbb{L}} fX \ for\ all\ X \subseteq P,$$

then f *is residuated.*

(2) *If* g *is a residual map from* $\mathbb{L}$ *to* $\mathbb{P}$*, then* g *preserves all existing infima in* $\mathbb{L}$*, that is, if* $t \in L$ *is the infimum (greatest lower bound) of* $Y \subseteq L$ *in* $\mathbb{L}$ *then* gt *is the infimum of* gY *in* $\mathbb{P}$.
In case $\mathbb{P}$ *and* $\mathbb{L}$ *are complete lattices, the reverse holds too: If a map* g *from* $\mathbb{L}$ *to* $\mathbb{P}$ *preserves all infima, that is,*

$$f(\inf_{\mathbb{P}} Y) = \inf_{\mathbb{L}} gY \ for\ all\ Y \subseteq L,$$

then g *is residual.*

(3) *For an adjunction* (f, g) *w.r.t.* $(\mathbb{P}, \mathbb{L})$ *the following hold:*
   (a1) f *is an isotone map from* $\mathbb{P}$ *to* $\mathbb{L}$.
   (a2) $f \circ g \circ f = f$
   (a3) fP *is a kernel system in* $\mathbb{L}$ *with* $f \circ g$ *as associated kernel operator on* $\mathbb{L}$*. In particular,* $L \to P, y \mapsto fgy$ *is a residual map from* $\mathbb{L}$ *to* $\mathbb{L}|fP$.
   (b1) g *is an isotone map from* $\mathbb{L}$ *to* $\mathbb{P}$.
   (b2) $g \circ f \circ g = g$
   (b3) gL *is a closure system in* $\mathbb{P}$ *with* $g \circ f$ *as associated closure operator on* $\mathbb{P}$*. In particular,* $P \to gL, x \mapsto gfx$ *is a residuated map from* $\mathbb{P}$ *to* $\mathbb{P}|gL$.

3 Adjunctions and Their Concept Posets

Definition 3. *Let* $\mathcal{P} := (\mathbb{P}, \mathbb{S}, \sigma, \sigma^+)$ *and* $\mathcal{Q} := (\mathbb{Q}, \mathbb{T}, \tau, \tau^+)$ *be poset adjunctions. Then a pair* (α, β) *forms a morphism from* $\mathcal{P}$ *to* $\mathcal{Q}$ *if* $(\mathbb{P}, \mathbb{Q}, \alpha, \alpha^+)$ *and* $(\mathbb{S}, \mathbb{T}, \beta, \beta^+)$ *are poset adjunctions satisfying*

$$\tau \circ \alpha = \beta \circ \sigma$$

Remark: This implies $\alpha^+ \circ \tau^+ = \sigma^+ \circ \beta^+$*, that is, the following diagrams are commutative:*

$$\begin{tikzcd} \mathbb{P} \arrow[r, "\alpha"] \arrow[d, "\sigma"'] & \mathbb{Q} \arrow[d, "\tau"] \\ \mathbb{S} \arrow[r, "\beta"'] & \mathbb{T} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{P} & \mathbb{Q} \arrow[l, "\alpha^+"'] \\ \mathbb{S} \arrow[u, "\sigma^+"] & \mathbb{T} \arrow[l, "\beta^+"] \arrow[u, "\tau^+"'] \end{tikzcd}$$

Next we illustrate the involved poset adjunctions:

$$\begin{tikzcd} \mathbb{P} \arrow[r, shift left, "\alpha"] \arrow[d, shift right, "\sigma"'] & \mathbb{Q} \arrow[l, shift left, "\alpha^+"] \arrow[d, shift right, "\tau"'] \\ \mathbb{S} \arrow[r, shift left, "\beta"] \arrow[u, shift right, "\sigma^+"'] & \mathbb{T} \arrow[l, shift left, "\beta^+"] \arrow[u, shift right, "\tau^+"'] \end{tikzcd}$$

Definition 4 (Concept Poset)**.** *For a poset adjunction* $\mathcal{P} = (\mathbb{P}, \mathbb{S}, \sigma, \sigma^+)$ *let*

$$\mathrm{B}\mathcal{P} := \{(p,s) \in P \times S \mid \sigma p = s \wedge \sigma^+ s = p\}$$

denote the set of (**formal**) **concepts** *in* $\mathcal{P}$*. Then the* **concept poset** *of* $\mathcal{P}$ *is given by*

$$\mathbb{B}\mathcal{P} := (\mathbb{P} \times \mathbb{S}) \mid \mathrm{B}\mathcal{P},$$

that is, $(p_0, s_0) \leqslant (p_1, s_1)$ *holds iff* $p_0 \leqslant p_1$ *iff* $s_0 \leqslant s_1$*, for all* $(p_0, s_0), (p_1, s_1) \in B\mathcal{P}$*. If* (p,s) *is a formal concept in* $\mathcal{P}$ *then* p *is referred to as* **extent** *in* $\mathcal{P}$ *and* s *as* **intent** *in* $\mathcal{P}$*.*

Theorem 1. *Let* (α, β) *be a morphism from a poset adjunction* $\mathcal{P} = (\mathbb{P}, \mathbb{S}, \sigma, \sigma^+)$ *to a poset adjunction* $\mathcal{Q} = (\mathbb{Q}, \mathbb{T}, \tau, \tau^+)$*. Then*

$$(\mathbb{B}\mathcal{P}, \mathbb{B}\mathcal{Q}, \Phi, \Psi)$$

is a poset adjunction for

$$\Phi : \mathrm{B}\mathcal{P} \to \mathrm{B}\mathcal{Q}, (p,s) \mapsto (\tau^+ \beta s, \beta s)$$

and

$$\Psi : \mathrm{B}\mathcal{Q} \to \mathrm{B}\mathcal{P}, (q,t) \mapsto (\alpha^+ q, \sigma \alpha^+ q).$$

In addition, if α *is surjective then so is* Φ*.*
Remark: In particular we want to point out that $\alpha^+ q$ *is an extent in* $\mathcal{P}$ *for every extent* q *in* $\mathcal{Q}$ *and similarly,* βs *is an intent in* $\mathcal{Q}$ *for every intent* s *in* $\mathcal{P}$*.*

Proof. Let $(p,s) \in \mathrm{B}\mathcal{P}$ and $(q,t) \in \mathrm{B}\mathcal{Q}$; then $\sigma p = s$ and $\sigma^+ s = p$ and $\tau q = t$ and $\tau^+ t = q$. This implies $\beta s = \beta\sigma p = \tau\alpha p$, thus

$$\Phi(p,s) = (\tau^+\beta s, \beta s) \in \mathrm{B}\mathcal{P}$$

(since $\tau\tau^+\beta s = \tau\tau^+\tau\alpha p = \tau\alpha p = \beta s$). Similarly, $\Psi(q,t) \in \mathrm{B}\mathcal{Q}$.
Assume now that $\Phi(p,s) \leqslant (q,t)$ holds, which implies $\beta s \leqslant t$. It follows that

$$\tau\alpha p = \beta\sigma p = \beta s \leqslant t$$

and hence

$$p \leqslant \alpha^+\tau^+ t = \alpha^+ q,$$

that is, $(p,s) \leqslant \Psi(q,t)$.
Conversely, assume that $(p,s) \leqslant \Psi(q,t)$ holds, which implies $p \leqslant \alpha^+ q$. It follows that

$$p \leqslant \alpha^+ q = \alpha^+\tau^+ t = \sigma^+\beta^+ t,$$

and hence $\beta s = \beta\sigma p \leqslant t$, that is, $\Phi(p,s) \leqslant (q,t)$.
Assume now that α is surjective; then $\alpha \circ \alpha^+ = \mathrm{id}_{\mathcal{Q}}$. Let $(q,t) \in \mathrm{B}\mathcal{P}$, that is, $\tau q = t$ and $\tau^+ t = q$. Then for $p := \alpha^+ q$ and $s := \sigma p$ we have $(p,s) \in \mathrm{B}\mathcal{P}$ since

$$\sigma^+ s = \sigma^+\sigma\alpha^+ q = \sigma^+\sigma\alpha^+\tau^+ t = \sigma^+\sigma\sigma^+\beta^+ t = \sigma^+\beta^+ t = \alpha^+\tau^+ t = \alpha^+ q = p.$$

Our claim is now that $\Phi(p,s) = (q,t)$ holds, that is, $\beta s = t$. The latter is true, since $\alpha p = \alpha\alpha^+ q = q$ implies

$$\beta s = \beta\sigma p = \tau\alpha p = \tau q = t.$$

□

Discussion for clarification: The question was raised whether, in the previous theorem, the residuated map Φ from $\mathrm{B}\mathcal{P}$ to $\mathrm{B}\mathcal{Q}$ allows some modification, since the map

$$P \times S \to Q \times T, (p,s) \mapsto (\alpha p, \beta s)$$

is obviously residuated from $\mathbb{P} \times \mathbb{S}$ to $\mathbb{Q} \times \mathbb{T}$. However, in general the latter map does not restrict to a map from $\mathrm{B}\mathcal{P}$ to $\mathrm{B}\mathcal{Q}$. Indeed, our construction of the map Φ is of the form $(p,s) \mapsto (\alpha' p, \beta s)$. As a warning, we want to point out that, in general, there is no residuated map from $\mathrm{B}\mathcal{P}$ to $\mathrm{B}\mathcal{Q}$ of the form $(p,s) \mapsto (\alpha p, \beta' s)$. The simple reason for this is that βs is an intent in $\mathcal{Q}$ for every intent s in $\mathcal{P}$, while there may exist an extent p in $\mathcal{P}$ such that αp is not an extent in $\mathcal{Q}$.

4 Morphisms between Pattern Structures

Definition 5. *A triple $\mathcal{G} = (G, \mathbb{D}, \delta)$ is a* **pattern setup** *if G is a set, $\mathbb{D} = (D, \sqsubseteq)$ is a poset, and $\delta : G \to D$ is a map. In case every subset of $\delta G := \{\delta g \mid g \in G\}$ has an infimum in $\mathbb{D}$, we will refer to $\mathcal{G}$ as* **pattern structure**. *Then the set*

$$\mathrm{C}_{\mathcal{G}} := \{\inf_{\mathbb{D}} \delta X \mid X \subseteq G\}$$

forms a closure system in $\mathbb{D}$.

If $\mathcal{G} = (G, \mathbb{D}, \delta)$ *and* $\mathcal{H} = (H, \mathbb{E}, \varepsilon)$ *each is a pattern setup, then a pair* (f, φ) *forms a* **pattern morphism** *from* $\mathcal{G}$ *to* $\mathcal{H}$ *if* $f : G \to H$ *is a map and* φ *is a residual map from* $\mathbb{D}$ *to* $\mathbb{E}$ *satisfying* $\varphi \circ \delta = \varepsilon \circ f$*, that is, the following diagram is commutative:*

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\delta"'] & H \arrow[d, "\varepsilon"] \\ \mathbb{D} \arrow[r, "\varphi"'] & \mathbb{E} \end{tikzcd}$$

In the sequel we show how our previous considerations apply to pattern structures.

Applications

(1) Let $\mathcal{G}$ be a pattern structure and $\mathcal{H}$ be a pattern setup. If (f, φ) is a pattern morphism from $\mathcal{G}$ to $\mathcal{H}$ with f being surjective, then $\mathcal{H}$ is also a pattern structure.

(2) Let $\mathcal{G} = (G, \mathbb{D}, \delta)$ and $\mathcal{H} = (H, \mathbb{E}, \varepsilon)$ be pattern structures. Also let (f, φ) be a pattern morphism from $\mathcal{G}$ to $\mathcal{H}$.
To apply the previous theorem we give the following construction:
f gives rise to an adjunction (α, α^+) between the power set lattices $\mathbf{2}^G := (2^G, \subseteq)$ and $\mathbf{2}^H := (2^H, \subseteq)$ via

$$\alpha : 2^G \to 2^H, X \mapsto fX$$

and

$$\alpha^+ : 2^H \to 2^G, Y \mapsto f^{-1}Y.$$

Further let φ^- denote the residuated map of φ w.r.t. $(\mathbb{E}, \mathbb{D})$, that is, $(\mathbb{E}, \mathbb{D}, \varphi^-, \varphi)$ is a poset adjunction. Then, obviously, $(\mathbb{D}^{\mathrm{op}}, \mathbb{E}^{\mathrm{op}}, \varphi, \varphi^-)$ is a poset adjunction too.

For pattern structures the following operators are essential:

$$\begin{aligned} {}^{\diamond} &: 2^G \to D, X \mapsto \inf_{\mathbb{D}} \delta X \\ {}^{\blacklozenge} &: D \to 2^G, d \mapsto \{g \in G \mid d \sqsubseteq \delta g\} \\ {}^{\square} &: 2^H \to E, Z \mapsto \inf_{\mathbb{E}} \varepsilon Z \\ {}^{\blacksquare} &: E \to 2^H, e \mapsto \{h \in H \mid e \sqsubseteq \varepsilon h\} \end{aligned}$$

It now follows that (α, φ) forms a morphism from the poset adjunction

$$\mathcal{P} = (\mathbf{2}^G, \mathbb{D}^{\mathrm{op}}, {}^{\diamond}, {}^{\blacklozenge})$$

to the poset adjunction

$$\mathcal{Q} = (\mathbf{2}^H, \mathbb{E}^{\mathrm{op}}, {}^{\square}, {}^{\blacksquare}).$$

In particular, $(fX)^{\square} = \varphi(X^{\diamond})$ holds for all $X \subseteq G$.
Here we give an illustration of the constructed adjunctions:

$$\begin{tikzcd}
\mathbf{2}^G \arrow[r, shift left, "\alpha"] \arrow[d, shift right, "\diamond"'] & \mathbf{2}^H \arrow[l, shift left, "\alpha^+"] \arrow[d, shift right, "\square"'] \\
\mathbb{D}^{\mathrm{op}} \arrow[r, shift left, "\varphi"] \arrow[u, shift right, "\blacklozenge"'] & \mathbb{E}^{\mathrm{op}} \arrow[l, shift left, "\varphi^-"] \arrow[u, shift right, "\blacksquare"']
\end{tikzcd}$$

Replacing $\mathbb{D}^{\mathrm{op}}$ by $\mathbb{D}$ and $\mathbb{E}^{\mathrm{op}}$ by $\mathbb{E}$ we receive the following commutative diagrams:

$$\begin{tikzcd}
\mathbf{2}^G \arrow[r, "\alpha"] \arrow[d, "\diamond"'] & \mathbf{2}^H \arrow[d, "\square"] \\
\mathbb{D} \arrow[r, "\varphi"'] & \mathbb{E}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{2}^G & \mathbf{2}^H \arrow[l, "\alpha^+"'] \\
\mathbb{D} \arrow[u, "\blacklozenge"] & \mathbb{E} \arrow[l, "\varphi^-"] \arrow[u, "\blacksquare"']
\end{tikzcd}$$

In combination we receive the following diagram of Galois connections and adjunctions between them:

$$\begin{tikzcd}
\mathbf{2}^G \arrow[r, shift left, "\alpha"] \arrow[d, shift right, "\diamond"'] & \mathbf{2}^H \arrow[l, shift left, "\alpha^+"] \arrow[d, shift right, "\square"'] \\
\mathbb{D} \arrow[r, shift left, "\varphi"] \arrow[u, shift right, "\blacklozenge"'] & \mathbb{E} \arrow[l, shift left, "\varphi^-"] \arrow[u, shift right, "\blacksquare"']
\end{tikzcd}$$

For the following we recollect that the **concept lattice** of $\mathcal{G}$ is given by $\mathbb{B}\mathcal{G} := \mathbb{B}\mathcal{P}$ — similarly, $\mathbb{B}\mathcal{H} := \mathbb{B}\mathcal{Q}$.
Now we are prepared to give an application of Theorem 1 to concept lattices of pattern structures: $(\mathbb{B}\mathcal{G}, \mathbb{B}\mathcal{H}, \Phi, \Psi)$ is an adjunction for

$$\Phi : \mathrm{B}\mathcal{G} \to \mathrm{B}\mathcal{H}, (X, d) \mapsto ((\varphi d)^{\blacksquare}, \varphi d)$$

and

$$\Psi : \mathrm{B}\mathcal{H} \to \mathrm{B}\mathcal{G}, (Z, e) \mapsto (f^{-1}Z, (f^{-1}Z)^{\diamond}).$$

In case f is surjective, Φ is surjective too.
Remark: This application implies a generalization of Proposition 1 in [2], that is, if Z is an extent in $\mathcal{H}$, then $f^{-1}Z$ is an extent in $\mathcal{G}$, and if d is an intent in $\mathcal{G}$ then φd is an intent in $\mathcal{H}$.

(3) Let $\mathcal{G} = (G, \mathbb{D}, \delta)$ be a pattern structure and let κ be a kernel operator on $\mathbb{D}$. Then $\varphi : D \to \kappa D, d \mapsto \kappa d$ forms a residual map from $\mathbb{D}$ to $\kappa\mathbb{D} := \mathbb{D} \mid \kappa D$, and (id_G, φ) is a pattern morphism from $\mathcal{G}$ to $\mathcal{H} := (G, \kappa\mathbb{D}, \varphi \circ \delta)$.
Remark: In [2], φ is called an o-projection. The above clarifies the role of o-projections for pattern structures.

(4) Let $\mathcal{G} = (G, \mathbb{D}, \delta)$ be a pattern structure, and let κ be a residual kernel operator on $\mathbb{D}$. Then (id_G, κ) is a pattern morphism from $\mathcal{G}$ to $\mathcal{H} := (G, \mathbb{D}, \kappa \circ \delta)$.
Remark: In [8], κ is also referred to as a residual projection. The above clarifies the role of residual projections for pattern structures.

(5) Generalizing [2] and [8], we observe that if $\mathcal{G} = (G, \mathbb{D}, \delta)$ is a pattern structure and φ is a residual map from $\mathbb{D}$ to $\mathbb{E}$, then (id_G, φ) is a pattern morphism from $\mathcal{G}$ to $\mathcal{H} = (G, \mathbb{E}, \varphi \circ \delta)$ satisfying that

$$\Phi : \mathrm{B}\mathcal{G} \to \mathrm{B}\mathcal{H}, (X, d) \mapsto ((\varphi d)^{\blacksquare}, \varphi d)$$

is a surjective residuated map from $\mathbb{B}\mathcal{G}$ to $\mathbb{B}\mathcal{H}$.
In particular, $X^{\square} = \varphi(X^{\diamond})$ holds for all $X \subseteq G$.
Remark: This application gives a better understanding to properly generalize the concept of projections as discussed in [3] and subsequently in [2, 4–8].

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