Attribute exploration with fuzzy attributes and background knowledge

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Abstract. Attribute exploration is a formal concept analytical tool for knowledge discovery by interactive determination of the implications holding between a given set of attributes. The corresponding algorithm queries the user in an efficient way about the implications between the attributes. The result of the exploration process is a representative set of examples for the entire theory and a set of implications from which all implications that hold between the considered attributes can be deduced. The method was successfully applied in different real-life applications for discrete data. In many instances, the user may know some implications before the exploration starts. These are considered as background knowledge and their usage shortens the exploration process. In this paper we show that the handling of background information can be generalised to the fuzzy setting.

Keywords: Knowledge discovery, Formal Concept Analysis, Fuzzy data

1 Introduction

Attribute exploration [1] allows the interactive determination of the implications holding between the attributes of a given context. However, there are situations when the object set of a context is too large (possibly infinite) or difficult to enumerate. With the examples (possibly none) of our knowledge we build the object set of the context step-by-step. The *stem base* of this context, that is a minimal set of non-redundant implications from which all the implications of the context follow, is built stepwise and we are asked whether the implications of the base are true. If an implication holds, then it is added to the stem base. If however, an implication does not hold, then we have to provide a counterexample. While performing an attribute exploration we have to be able to distinguish between true and false implications and to provide correct counterexamples for false implications. This is a crucial point since the algorithm is naive and will believe whatever we tell it. Once a decision was taken about the validity of an implication, the choice cannot be reversed. Therefore, the counterexamples may not contradict the so-far confirmed implications. The procedure ends when all implications of the current stem base hold in general. This way we obtain an object set which is representative for the entire theory, that may also be infinite.

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The exploration process can be shortened by taking some *background knowledge* [2] into account that the user has at the beginning of the exploration.

As many data sets contain fuzzy data, it is a natural wish to generalise attribute exploration to the fuzzy setting. In [3] we have already shown how this can be done. It turned out that we have to make some restrictions on the implications (using the globalisation as the hedge) in order to be able to perform a successful attribute exploration. In this paper we generalise the fuzzy attribute exploration to the case with background knowledge. The main work starts in Section 3 where we show how non-redundant bases can be obtained while using background knowledge. The theory for the exploration is developed in Section 4 including an appropriate algorithm. In Section 5 we use a real-world data set to illustrative both exploration with and without background knowledge.

It should be mentioned that there is some overlap with results presented in the authors PhD thesis [4], see Chapter 5.

2 Preliminaries

2.1 Crisp Attribute Exploration

We assume basic familiarities with Formal Concept Analysis [1].

In the introductory section we have already explained the principle of attribute exploration. However, we have not yet presented its key to success. This is, why we do not have to reconsider already confirmed implications after adding new objects to the context.

Proposition 1. *([1]) Let $\mathbb{K}$ be a context and $P_1, P_2, \ldots, P_n$ be the first n pseudo-intents of $\mathbb{K}$ with respect to the lectic order. If $\mathbb{K}$ is extended by an object g the object intent $g^{\uparrow}$ of which respects the implications $P_i \to P_i^{\downarrow\uparrow}$, $i \in \{1, \ldots, n\}$, then $P_1, P_2, \ldots, P_n$ are also the lectically first n pseudo-intents of the extended context.*

Attribute exploration was successfully applied in both theoretical and practical research domains. On the one hand it facilitated the discovery of implications between properties of mathematical structures, see for example [5–7]. On the other hand it was also used in real-life scenarios, for instance in chemistry [8], information systems [9], etc.

In case the user knows some implications between the attributes in advance, *attribute exploration with background knowledge* [10, 2] can be applied. Using background knowledge in the exploration will considerably reduce the time of the process as the user has to answer less questions and provide less counterexamples.

2.2 Formal Fuzzy Concept Analysis

Due to lack of space we omit the introduction of some basic notions from fuzzy logic, fuzzy sets and Fuzzy Formal Concept Analysis. We assume that the reader is familiar with these notions and refer to standard literature. For fuzzy theory we refer to [11, 12], in particular, notions like residuated lattices with hedges and **L**-sets can be found for instance in [13]. For Fuzzy Formal Concept Analysis see [14, 15] but also [16, 17]. The theory about Fuzzy Formal Concept Analysis with hedges can be found in [13]. In this section we only present some notions concerning lectic order, attribute implications and the computation of their non-redundant bases.

We start with the *fuzzy lectic order* [18] which is defined as follows: Let $L = \{l_0 < l_1 < \cdots < l_n = 1\}$ be the support set of some linearly ordered residuated lattice and $M = \{1, 2, \ldots, m\}$ the attribute set of an **L**-context (G, M, I). For $(x, i), (y, j) \in M \times L$, we write

$$(x, l_i) \leq (y, l_j) :\Longleftrightarrow (x < y) \text{ or } (x = y \text{ and } l_i \geq l_j).$$

We define $B \oplus (x, i) := ((B \cap \{1, 2, \ldots, x-1\}) \cup \{{}^{l_i}/x\})^{\downarrow\uparrow}$ for $B \in \mathbf{L}^M$ and $(x, i) \in M \times L$. Furthermore, for $B, C \in \mathbf{L}^M$, we define $B <_{(x,i)} C$ by

$$B \cap \{1, \ldots, x-1\} = C \cap \{1, \ldots, x-1\} \text{ and } B(x) < C(x) = l_i. \qquad (1)$$

We say that B is **lectically smaller** than C, written $B < C$, if $B <_{(x,i)} C$ for some (x, i) satisfying (1). As in the crisp case, we have that $B^+ := B \oplus (x, i)$ is the least intent which is greater than a given B with respect to $<$ and (x, i) is the greatest with $B <_{(x,i)} B \oplus (x, i)$ (for details we refer to [18]).

Fuzzy Implications and Non-redundant Bases. Fuzzy attribute implications were studied in a series of papers by Bělohlávek and Vychodil [19, 20].

We denote by $\mathrm{S}(A, B)$ the truth value of "the **L**-set A is a subset of the **L**-set B". Futher, $(-)^*$ denotes the hedge of a residuated lattice **L**, i.e., $(-)^* : L \to L$ is a map satisfying $a^* \leq a$, $(a \to b)^* \leq a^* \to b^*$, $a^{**} = a^*$ and $1^* = 1$ for every $a, b \in L$. Typical examples for the hedge are the *identity*, i.e., $a^* := a$ for all $a \in L$, and the *globalisation*, i.e., $a^* := 0$ for all $a \in L \setminus \{1\}$ and $a^* := 1$ if and only if $a = 1$.

A **fuzzy attribute implication** (over the attribute set M) is an expression $A \Rightarrow B$, where $A, B \in \mathbf{L}^M$. The verbal meaning of $A \Rightarrow B$ is: "if it is (very) true that an object has all attributes from A, then it also has all attributes from B". The notions "being very true", "to have an attribute", and the logical connective "if-then" are determined by the chosen **L**. For an **L**-set $N \in \mathbf{L}^M$ of attributes, the degree $||A \Rightarrow B||_N \in L$ to which $A \Rightarrow B$ is valid in N is defined as

$$||A \Rightarrow B||_N := S(A, N)^* \to S(B, N).$$

If N is the **L**-set of all attributes of an object g, then $||A \Rightarrow B||_N$ is the truth degree to which $A \Rightarrow B$ holds for g. For $\mathcal{N} \subseteq \mathbf{L}^M$, the degree $||A \Rightarrow B||_{\mathcal{N}} \in L$ to which $A \Rightarrow B$ holds in $\mathcal{N}$ is defined by $||A \Rightarrow B||_{\mathcal{N}} := \bigwedge_{N \in \mathcal{N}} ||A \Rightarrow B||_N$. For an **L**-context (G, M, I), let $I_g \in \mathbf{L}^M$ $(g \in G)$ be an **L**-set of attributes such that $I_g(m) = I(g, m)$ for each $m \in M$. Clearly, I_g corresponds to the row labelled g in (G, M, I). We define the degree $||A \Rightarrow B||_{(G,M,I)} \in L$ to which $A \Rightarrow B$ holds in (each row of) $\mathbb{K} = (G, M, I)$ by $||A \Rightarrow B||_{\mathbb{K}} := ||A \Rightarrow B||_{\mathcal{N}}$, where

$\mathcal{N} := \{I_g \mid g \in G\}$. Denote by $\mathrm{Int}(G^*, M, I)$ the set of all intents of $\mathfrak{B}(G^*, M, I)$. The degree $||A \Rightarrow B||_{\mathfrak{B}(G^*,M,I)} \in L$ to which $A \Rightarrow B$ holds in (the intents of) $\mathfrak{B}(G^*, M, I)$ is defined by

$$||A \Rightarrow B||_{\mathfrak{B}(G^*,M,I)} := ||A \Rightarrow B||_{\mathrm{Int}(G^*,M,I)}. \tag{2}$$

Lemma 1. *([20]) For each fuzzy attribute implication $A \Rightarrow B$ it holds that $||A \Rightarrow B||_{(G,M,I)} = ||A \Rightarrow B||_{\mathfrak{B}(G^*,M,I)} = S(B, A^{\downarrow\uparrow})$.*

Due to the large number of implications in a formal context, one is interested in the stem base of the implications. Neither the existence nor the uniqueness of the stem base for a given **L**-context are guaranteed in general [20].

Let T be a set of fuzzy attribute implications. An **L**-set of attributes $N \in \mathbf{L}^M$ is called a **model** of T if $||A \Rightarrow B||_N = 1$ for each $A \Rightarrow B \in T$. The set of all models of T is denoted by $\mathrm{Mod}(T) := \{N \in \mathbf{L}^M \mid N \text{ is a model of } T\}$. The degree $||A \Rightarrow B||_T \in L$ to which $A \Rightarrow B$ **semantically follows** from T is defined by $||A \Rightarrow B||_T := ||A \Rightarrow B||_{\mathrm{Mod}(T)}$. T is called **complete** in (G, M, I) if $||A \Rightarrow B||_T = ||A \Rightarrow B||_{(G,M,I)}$ for each $A \Rightarrow B$. If T is complete and no proper subset of T is complete, then T is called a **non-redundant basis**.

Theorem 1. *([20]) T is complete iff $\mathrm{Mod}(T) = \mathrm{Int}(G^*, M, I)$.*

As in the crisp case the stem base of a given **L**-context can be obtained through the pseudo-intents. $\mathcal{P} \subseteq \mathbf{L}^M$ is called a **system of pseudo-intents** if for each $P \in \mathbf{L}^M$ we have:

$$P \in \mathcal{P} \iff (P \neq P^{\downarrow\uparrow} \text{ and } ||Q \Rightarrow Q^{\downarrow\uparrow}||_P = 1 \text{ for each } Q \in \mathcal{P} \text{ with } Q \neq P).$$

Theorem 2. *([20]) $T := \{P \Rightarrow P^{\downarrow\uparrow} \mid P \in \mathcal{P}\}$ is complete and non-redundant. If $(-)^*$ is the globalisation, then T is unique and minimal.*

3 Adding background knowledge to the stem base

The user may know some implications between attributes in advance. We will call such kind of implications *background implications*. In this section we will focus on finding a minimal list of implications, which together with the background implications will describe the structure of the concept lattice.

The theory about background knowledge for the crisp case was developed in [2] and a more general form of it in [10]. The results from [2] for implication bases with background knowledge follow by some slight modifications of the results about implication bases without background knowledge presented in [1]. The same applies for the fuzzy variant of this method. Hence, if we choose the empty set as the background knowledge, we obtain the results from [19, 20].

We start by investigating the stem bases of **L**-contexts relative to a set of background implications. Afterwards we show how some notions and results for fuzzy implications and their stem bases change for our new setting.

The attribute sets of **L**-contexts and the residuated lattices **L** will be considered finite. Further, **L** is assumed to be linearly ordered.

Definition 1. *Let $\mathbb{K}$ be a finite $\mathbf{L}$-context and let $\mathcal{L}$ be a set of background attribute implications. A set $\mathcal{B}$ of fuzzy attribute implications of $\mathbb{K}$ is called **$\mathcal{L}$-complete** if every implication of $\mathbb{K}$ is entailed by $\mathcal{L} \cup \mathcal{B}$. We call $\mathcal{B}$, **$\mathcal{L}$-non-redundant** if no implication $A \Rightarrow B$ from $\mathcal{B}$ is entailed by $(\mathcal{B} \setminus \{A \Rightarrow B\}) \cup \mathcal{L}$. If $\mathcal{B}$ is both $\mathcal{L}$-complete and $\mathcal{L}$-non-redundant, it is called an **$\mathcal{L}$-base**.*

Note that if we have $\mathcal{L} = \varnothing$ in the above definition, then all $\mathcal{L}$-notions are actually the notions introduced for sets of fuzzy implications. This remark holds also for the other notions introduced in this section.

For any set $\mathcal{L}$ of background attribute implications and any attribute $\mathbf{L}$-set $X \in \mathbf{L}^M$ we define an $\mathbf{L}$-set $X^{\mathcal{L}} \in \mathbf{L}^M$ and an $\mathbf{L}$-set $X^{\mathcal{L}_n} \in \mathbf{L}^M$ for each non-negative integer n by

$$X^{\mathcal{L}} := X \cup \bigcup \{B \otimes \mathrm{S}(A, X)^* \mid A \Rightarrow B \in \mathcal{L}\},$$
$$X^{\mathcal{L}_n} := \begin{cases} X, & n = 0, \\ (X^{\mathcal{L}_{n-1}})^{\mathcal{L}}, & n \geq 1. \end{cases}$$

An operator $\mathcal{L}$ on these sets is defined by

$$\mathcal{L}(X) := \bigcup_{n=0}^{\infty} X^{\mathcal{L}_n}. \tag{3}$$

From [20] we know that $\mathcal{L}(-)$ is an $\mathbf{L}^*$-closure operator for a finite set M of attributes and a finite residuated lattice $\mathbf{L}$.

Definition 2. *For an $\mathbf{L}$-context (G, M, I), a subset $\mathcal{P} \subseteq \mathbf{L}^M$ is called a **system of $\mathcal{L}$-pseudo-intents** of (G, M, I) if for each $P \in \mathbf{L}^M$ the following holds*

$$P \in \mathcal{P} \iff (P = \mathcal{L}(P) \neq P^{\downarrow\uparrow} \text{ and } ||Q \Rightarrow Q^{\downarrow\uparrow}||_P = 1 \text{ for each } Q \in \mathcal{P} : Q \neq P).$$

As in the case without background knowledge, the usage of the globalisation simplifies the definition of the system of $\mathcal{L}$-pseudo-intents. We have that $\mathcal{P} \subseteq \mathbf{L}^M$ is a system of pseudo-intents if

$$P \in \mathcal{P} \iff (P = \mathcal{L}(P) \neq P^{\downarrow\uparrow} \text{ and } Q^{\downarrow\uparrow} \subseteq P \text{ for each } Q \in \mathcal{P} \text{ with } Q \subsetneq P).$$

Theorem 3. *The set $\mathcal{B}_{\mathcal{L}} := \{P \Rightarrow P^{\downarrow\uparrow} \mid P$ is an $\mathcal{L}$-pseudo-intent$\}$ is an $\mathcal{L}$-base of $\mathbb{K}$. We call it the **$\mathcal{L}$-Duquenne-Guigues-base** or the **$\mathcal{L}$-stem base**.*

Proof. First note that all implications from $\mathcal{B}_{\mathcal{L}}$ are implications of (G, M, I). We start by showing that $\mathcal{B}_{\mathcal{L}}$ is complete, i.e., $||A \Rightarrow B||_{\mathcal{B}_{\mathcal{L}} \cup \mathcal{L}} = ||A \Rightarrow B||_{(G,M,I)}$ for every fuzzy implication $A \Rightarrow B$. By Equation (2) follows $||A \Rightarrow B||_{(G,M,I)} = ||A \Rightarrow B||_{\mathrm{Int}(G^*,M,I)}$. Hence, it is sufficient to prove that $||A \Rightarrow B||_{\mathcal{B}_{\mathcal{L}} \cup \mathcal{L}} = ||A \Rightarrow B||_{\mathrm{Int}(G^*,M,I)}$ for every fuzzy attribute implication $A \Rightarrow B$. For any $\mathbf{L}$-set $N \in \mathbf{L}^M$, $N \Rightarrow \mathcal{L}(N)$ is entailed by $\mathcal{L}$, therefore we have $N = \mathcal{L}(N)$.

Each intent $N \in \mathrm{Int}(G^*, M, I)$ is a model of $\mathcal{B}_{\mathcal{L}}$. Now let $N \in \mathrm{Mod}(\mathcal{B}_{\mathcal{L}})$ and assume that $N \neq N^{\downarrow\uparrow}$, i.e., N is not an intent. Since $N \in \mathrm{Mod}(\mathcal{B}_{\mathcal{L}})$ we have

$||Q \Rightarrow Q^{\downarrow\uparrow}||_N = 1$ for every $\mathcal{L}$-pseudo-intent $Q \in \mathcal{P}$. By definition, N is an $\mathcal{L}$-pseudo-intent and hence $N \Rightarrow N^{\downarrow\uparrow}$ belongs to $\mathcal{B}_\mathcal{L}$. But now, we have

$$||N \Rightarrow N^{\downarrow\uparrow}||_N = \mathrm{S}(N,N)^* \to \mathrm{S}(N^{\downarrow\uparrow},N) = 1^* \to \mathrm{S}(N^{\downarrow\uparrow},N) = \mathrm{S}(N^{\downarrow\uparrow},N) \neq 1,$$

which is a contradiction because N does not respect this implication.

To finish the proof, we still have to show that $\mathcal{B}_\mathcal{L}$ is $\mathcal{L}$-non-redundant. To this end let $P \Rightarrow P^{\downarrow\uparrow} \in \mathcal{B}_\mathcal{L}$. We show that this implication is not entailed by $\underline{\mathcal{L}} := (\mathcal{B}_\mathcal{L} \setminus \{P \Rightarrow P^{\downarrow\uparrow}\}) \cup \mathcal{L}$. As $P = \mathcal{L}(P)$, it is obviously a model of $\mathcal{L}$. We have $||Q \Rightarrow Q||_P = 1$ for every $\mathcal{L}$-pseudo-intent $Q \in \mathcal{P}$ different from P since P is an $\mathcal{L}$-pseudo-intent. Therefore, $P \in \mathrm{Mod}(\underline{\mathcal{L}})$. We also have that $||P \Rightarrow P^{\downarrow\uparrow}||_P = \mathrm{S}(P^{\downarrow\uparrow},P) \neq 1$ and thus P is not a model of $\mathcal{B}_\mathcal{L} \cup \mathcal{L}$. Hence, we have $||P \Rightarrow P^{\downarrow\uparrow}||_{(G,M,I)} = ||P \Rightarrow P^{\downarrow\uparrow}||_{\mathcal{B}_\mathcal{L} \cup \mathcal{L}} \neq ||P \Rightarrow P^{\downarrow\uparrow}||_{\underline{\mathcal{L}}}$, showing that $\underline{\mathcal{L}}$ is not complete and thus $\mathcal{B}_\mathcal{L} \cup \mathcal{L}$ is non-redundant. $\square$

In general we write $P \Rightarrow P^{\downarrow\uparrow} \setminus \{m \in M \mid P(m) = P^{\downarrow\uparrow}(m)\}$ instead of $P \Rightarrow P^{\downarrow\uparrow}$.

Note that computing the stem-base and closing the implications from it with respect to the operator $\mathcal{L}(-)$ will yield a different set of implications than the $\mathcal{L}$-stem-base. Let us take a look at the following example.

Example 1. Consider the **L**-context given in Figure 1. In order to ensure that its stem-base and $\mathcal{L}$-stem-base exist, we use the globalisation. Further, we use the Gödel logic. The stem-base is displayed in the left column of Figure 1. For the background implications $\mathcal{L} := \{b \Rightarrow a,\ d \Rightarrow a,\ \{a,c\} \Rightarrow b\}$ we obtain the $\mathcal{L}$-stem-base displayed in the middle column of the figure. If we close the pseudo-intents

	a	b	c	d
x	1	0.5	0	0
y	1	0	0	0
z	0	0	1	0
t	0	0	0	0.5

stem base	$\mathcal{L}$-stem base	$\mathcal{L}(P) \Rightarrow P^{\downarrow\uparrow}$
${}^{0.5}/b \Rightarrow a,$	${}^{0.5}/b \Rightarrow a,$	${}^{0.5}/b \Rightarrow a,$
${}^{0.5}/a \Rightarrow a,$	${}^{0.5}/a \Rightarrow a,$	${}^{0.5}/a \Rightarrow a,$
$d \Rightarrow a,b,c,$	$c, {}^{0.5}/d \Rightarrow a,b,d,$	$a,d \Rightarrow b,c,$
$c, {}^{0.5}/d \Rightarrow a,b,d,$	$a, {}^{0.5}/d \Rightarrow b,c,d,$	$c, {}^{0.5}/d \Rightarrow a,b,d,$
$b \Rightarrow a,c,d,$	$a,d \Rightarrow b,c,$	$a,b \Rightarrow c,d,$
$a, {}^{0.5}/d \Rightarrow b,c,d,$	$a,b \Rightarrow c,d.$	$a, {}^{0.5}/d \Rightarrow b,c,d,$
$a,c \Rightarrow b,d.$		$a,b,c \Rightarrow d.$

Fig. 1. An **L**-context and its different stem bases.

of the stem-base with respect to the operator $\mathcal{L}(-)$, we obtain implications of the form $\mathcal{L}(P) \Rightarrow P^{\downarrow\uparrow}$, which are displayed in the right column of the figure. As one easily sees, the latter set of implications and the $\mathcal{L}$-stem-base are different. The set $\{\mathcal{L}(P) \Rightarrow P^{\downarrow\uparrow} \mid P \text{ is a pseudo-intent with } \mathcal{L}(P) \neq P^{\downarrow\uparrow}\}$ contains an additional implication, namely $\{a,b,c\} \Rightarrow d$.

For developing our theory about fuzzy attribute exploration with background knowledge, the following results are useful. First, the set of all intents and all

$\mathcal{L}$-pseudo-intents is an $\mathbf{L}^*$-closure system, as stated below. Due to lack of space we omit the proofs of the following two lemmas.

Lemma 2. *Let (G, M, I) be an $\mathbf{L}$-context, let $\mathcal{L}$ be a set of fuzzy implications of (G, M, I). Further, let P and Q be intents or $\mathcal{L}$-pseudo-intents such that*

$$\mathrm{S}(P, Q)^* \leq \mathrm{S}(P^{\downarrow\uparrow}, P \cap Q) \quad \textit{and} \quad \mathrm{S}(Q, P)^* \leq \mathrm{S}(Q^{\downarrow\uparrow}, P \cap Q).$$

Then, $P \cap Q$ is an intent.

Note that if we choose the globalisation for $(-)^*$, then $P \cap Q$ is an intent provided that P and Q are ($\mathcal{L}$-pseudo-)intents with $P \nsubseteq Q$ and $Q \nsubseteq P$.

Now we are interested in the closure of an $\mathbf{L}$-set with respect to the implications of the $\mathcal{L}$-base $\mathcal{B}_{\mathcal{L}}$. Therefore, we first define for each $\mathbf{L}$-set $X \in \mathbf{L}^M$ and each non-negative integer n the $\mathbf{L}$-sets $X^{\mathcal{L}^\bullet}, X^{\mathcal{L}^\bullet_n} \in \mathbf{L}^M$ as follows:

$$X^{\mathcal{L}^\bullet} := X \cup \bigcup \{B \otimes \mathrm{S}(A, X)^* \mid A \Rightarrow B \in \mathcal{B}_{\mathcal{L}}, A \neq X\},$$
$$X^{\mathcal{L}^\bullet_n} := \begin{cases} X, & n = 0, \\ (X^{\mathcal{L}^\bullet_{n-1}})^{\mathcal{L}^\bullet}, & n \geq 1. \end{cases}$$

Further, we define an operator $\mathcal{L}^\bullet(-)$ on these sets by

$$\mathcal{L}^\bullet(X) := \bigcup_{n=0}^{\infty} X^{\mathcal{L}^\bullet_n}. \tag{4}$$

Lemma 3. *If $(-)^*$ is the globalisation, then $\mathcal{L}^\bullet$ given by (4) is an $\mathbf{L}^*$-closure operator and $\{\mathcal{L}^\bullet(X) \mid X \in \mathbf{L}^M\}$ coincides with the set of all $\mathcal{L}$-pseudo-intens and intents of (G, M, I).*

Remark 1. Note that for a general hedge, $\mathcal{L}^\bullet(-)$ does not need be an $\mathbf{L}^*$-closure operator. For instance, choose the Goguen structure and the identity for the hedge $(-)^*$. Further, let $\mathcal{L} := \{{}^{0.3}/y \Rightarrow y\}$. Then,

$$\begin{aligned} \mathcal{L}^\bullet(\{{}^{0.2}/y\})(y) \geq (\{{}^{0.2}/y\})^{\mathcal{L}^\bullet}(y) &= \{{}^{0.2}/y\} \cup \{y \otimes (0.3 \to 0.2)\} \\ &= \{{}^{0.2}/y\} \cup \{{}^{0.(66)}/y\} = \{{}^{0.(66)}/y\}, \end{aligned}$$

and $\mathcal{L}^\bullet(\{{}^{0.3}/y\})(y) = \{{}^{0.3}/y\}$. Hence, $\mathcal{L}^\bullet(-)$ does not satisfy the monotony property, because we have $\{{}^{0.2}/y\} \subseteq \{{}^{0.3}/y\}$ but $\mathcal{L}^\bullet(\{{}^{0.2}/y\}) \nsubseteq \mathcal{L}^\bullet(\{{}^{0.3}/y\})$.

4 Attribute exploration with background knowledge

Particularly appealing is the usage of background knowledge in the exploration process. This proves itself to be very useful and time saving for the user. He/she will have to answer less questions, as the algorithm does not start from scratch.

Due to Remark 1 and the fact that we are only able to perform a successful attribute exploration if the chosen hedge is the globalisation, we will consider only this hedge in this section. In order to arrive at the exploration with background knowledge we will present the lectic order, the "key proposition" and an appropriate algorithm for attribute exploration in this setting.

The lectic order is defined analogously as in Section 2, see (1). The only difference lies in the definition of "$\oplus$". This time we are using the $\mathbf{L}^*$-closure operator $(-)^{\mathcal{L}^\bullet}$ instead of $(-)^{\downarrow\uparrow}$.

Theorem 4. *The lectically first intent or $\mathcal{L}$-pseudo-intent is $\varnothing^{\mathcal{L}^\bullet}$. For a given $\mathbf{L}$-set $A \in \mathbf{L}^M$, the lectically next intent or $\mathcal{L}$-pseudo-intent is given by the $\mathbf{L}$-set $A \oplus (m,l)$, where $(m,l) \in M \times L$ is the greatest tuple such that $A <_{(m,l)} A \oplus (m,l)$. The lectically last intent or $\mathcal{L}$-pseudo-intent is M.*

Now we are prepared to present the main proposition regarding attribute exploration with background knowledge in a fuzzy setting.

Proposition 2. *Let $\mathbf{L}$ be a finite, linearly ordered residuated lattice with globalisation. Further, let $\mathcal{P}$ be the unique system of $\mathcal{L}$-pseudo-intents of a finite $\mathbf{L}$-context $\mathbb{K}$ with $P_1, \ldots, P_n \in \mathcal{P}$ being the first n $\mathcal{L}$-pseudo-intents in $\mathcal{P}$ with respect to the lectic order. If $\mathbb{K}$ is extended by an object g, the object intent $g^\uparrow$ of which respects the implications from $\mathcal{L} \cup \{P_i \Rightarrow P_i^{\downarrow\uparrow} \mid i \in \{1, \ldots, n\}\}$, then $P_1, \ldots, P_n$ remain the lectically first n $\mathcal{L}$-pseudo-intents of the extended context.*

Proof. Let $\mathbb{K} = (H, M, J)$ be the initial context and let (G, M, I) be the extended context, namely $G = H \cup \{g\}$ and $J = I \cap (H \times M)$. To put it briefly, since g^I is a model of $P_i \Rightarrow P_i^{JJ}$ for all $i \in \{1, \ldots, n\}$ we have that $P_i^{JJ} = P_i^{II}$. By the definition of $\mathcal{L}$-pseudo-intents and the fact that every $\mathcal{L}$-pseudo-intent Q of (H, M, J) with $Q \subset P_i$ is lectically smaller than P_i, we have that $P_1, \ldots, P_n$ are the lectically first n $\mathcal{L}$-pseudo-intents of (G, M, I).

We now have the key to a successful attribute exploration with background knowledge in the fuzzy setting, at least when we use the globalisation. With this result we are able to generalise the attribute exploration algorithm as presented by Algorithm 1. Its input is the $\mathbf{L}$-context $\mathbb{K}$, the residuated lattice $\mathbf{L}$ and the set of background implications $\mathcal{L}$. The first intent or $\mathcal{L}$-pseudo-intent is the empty set. If it is an intent, add it to the set of intents of the context. Otherwise, ask the expert whether the implication is true in general. If so, add this implication to the $\mathcal{L}$-stem base, otherwise ask for a counterexample and add it to the context (line $2-11$). Until A is different from M, repeat the following steps: Search for the largest attribute i in M with its largest value l such that $A(i) < l$. For this attribute compute its closure with respect to the $\mathcal{L}^\bullet(-)$-closure operator given by (4) and check whether the result is the lectically next intent or $\mathcal{L}$-pseudo-intent (line $12-16$). Thereby, $A \searrow i := A \cap \{1, \ldots, i-1\}$. In lines $17-25$ we repeat the same procedure as in lines $2-11$.

The algorithm generates interactively the $\mathcal{L}$-stem base of the $\mathbf{L}$-context. We enumerate the intents and pseudo-intents in the lectic order. Due to Proposition 2 we are allowed to extend the context by objects whose object intents respect the already confirmed implications. This way, the $\mathcal{L}$-pseudo-intents already contained in the stem base do not change. Hence, the algorithm is sound and correct.

Algorithm 1: FuzzyExploration($\mathbb{K}, \mathbf{L}, \mathcal{L}$)

```
$\mathcal{L} := \varnothing$; $A := \varnothing$;
if $A = A^{\downarrow\uparrow}$ then
    add $A$ to Int($\mathbb{K}$)
else
    Ask expert whether $A \Rightarrow A^{\downarrow\uparrow}$ is valid;
    if yes then
        add $A \Rightarrow A^{\downarrow\uparrow}$ to $\mathcal{L}$
    else
        Ask for counterexample $g$ and add it to $\mathbb{K}$
    end
end
while $A \neq M$ do
    for $i = n, \ldots, 1$ and $l = \max L, \ldots, \min L$ with $A(i) < l$ do
        $B := \mathcal{L}^{\bullet}(A)$;
        if $A \searrow i = B \searrow i$ and $A(i) < B(i)$ then
            $A := B$;
            if $A = A^{\downarrow\uparrow}$ then
                add $A$ to Int($\mathbb{K}$)
            else
                Ask expert whether $A \Rightarrow A^{\downarrow\uparrow}$ is valid;
                if yes then
                    add $A \Rightarrow A^{\downarrow\uparrow}$ to $\mathcal{L}$
                else
                    Ask for counterexample $g$ and add it to $\mathbb{K}$
                end
            end
        end
    end
end
```

5 Illustrative example

For our illustrative example we will consider the data from Figure 2. The objects are different universities from Germany and the attributes are indicators rating these institutions. **Study situation overall**: M.Sc. students were asked about their overall rating of their studies. **IT-infrastructure**: the availability of subject-specific software, PC pools and wifi were taken into account. **Courses offered**: amount of courses and the interdisciplinary references were relevant. **Possibility of studying**: timing of the courses and content of the curriculum were decisive. **Passage to M.Sc.**: clearly formulated admission requirements and assistance of the students with the organisational issues were relevant.

Suppose we want to explore the implications between the attributes from the **L**-context from Figure 2. We also know some examples, namely the TUs. These will be the objects of the **L**-context we start with. Further, we heard from others that the implications $\{a, b\} \Rightarrow c$, $d \Rightarrow \{^{0.5}/a, e\}$, $\{a, e\} \Rightarrow \{c, d\}$ and $a \Rightarrow \{^{0.5}/b, {}^{0.5}/c, {}^{0.5}/e\}$ hold. These will be considered the set of background implications $\mathcal{L}$. The exploration process is displayed in the left column of Figure 3. The first $\mathcal{L}$-pseudo-intent is $\varnothing$ and we ask the expert whether $\varnothing \Rightarrow \varnothing^{\downarrow\uparrow}$ holds. This is not the case and a counterexample is Uni Bochum. For implications that hold, for instance in step no. 3, the expert answers just “yes” and the implication is added to the $\mathcal{L}$-base. Afterwards, the validity of the implication induced by the next $\mathcal{L}$-pseudo-intent is asked, and so on. The algorithm continues until we reach M as an intent or $\mathcal{L}$-pseudo-intent. In our case, however, the algorithm stops at step no. 10. This is due to the fact that the implications induced by the $\mathcal{L}$-pseudo-intents after $\{b, {}^{0.5}/d\}$ are confirmed by the implications from the background knowledge. The result of the exploration is an extended context, namely

	study situation overall a	IT- infra-structure b	courses offered c	possibility of studying d	passage to M.Sc. e
TU Braunschweig	0.5	0	0.5	0.5	0
TU Chemnitz	0.5	1	0.5	0.5	0.5
TU Clausthal	1	1	1	1	1
TU Darmstadt	0.5	0.5	0.5	0.5	0.5
TU Ilmenau	0.5	1	0.5	0.5	1
TU Kaiserslautern	1	0.5	0.5	0.5	0
Uni Bielefeld	0.5	0	0.5	1	1
Uni Bochum	0	0.5	0.5	0.5	1
Uni Duisburg	0.5	0.5	0	0.5	0.5
Uni Erlangen	0.5	0.5	0.5	0.5	0
Uni Heidelberg	0.5	1	0.5	1	1
Uni Koblenz	0.5	0.5	0.5	0.5	0.5
Uni Saarbrücken	1	0.5	1	1	1

Fig. 2. The data is an extract from the data published in the journal Zeit Campus in January 2013. The whole data can be found under https://bit.ly/ZCinsr-informatik.

that contains our initial examples and the counterexamples we (or the expert) has entered. We also obtain an $\mathcal{L}$-base consisting of the background implications $\mathcal{L}$ and the implications we have confirmed during the exploration process. In the right column of Figure 3 the exploration without background knowledge is displayed. One immediately sees that there are 4 additional steps. The first difference between the explorations appears in step no. 8. Without using background knowledge we have to answer the implication from the right column, while this implication is already confirmed if we use background knowledge. The exploration without background knowledge yields the same extended context whereas the stem base contains the implications we have confirmed during the process.

no.	expl. w. background know.	simple expl.
1	Q: $\{\} \Rightarrow \{^{0.5}/a, {}^{0.5}/c, {}^{0.5}/d\}$ E: no, ex. Uni Bochum	Q: $\{\} \Rightarrow \{^{0.5}/a, {}^{0.5}/c, {}^{0.5}/d\}$ E: no, ex. Uni Bochum
2	Q: $\{\} \Rightarrow \{^{0.5}/c, {}^{0.5}/d\}$ E: no, ex. Uni Duisburg	Q: $\{\} \Rightarrow \{^{0.5}/c, {}^{0.5}/d\}$ E: no, ex. Uni Duisburg
3	Q: $\{\} \Rightarrow {}^{0.5}/d$ E: yes	Q: $\{\} \Rightarrow {}^{0.5}/d$ E: yes
4	Q: $\{^{0.5}/d, {}^{0.5}/e\} \Rightarrow {}^{0.5}/b$ E: no, ex. Uni Bielefeld	Q: $\{^{0.5}/d, {}^{0.5}/e\} \Rightarrow {}^{0.5}/b$ E: no, ex. Uni Bielefeld
5	Q: $\{^{0.5}/b, {}^{0.5}/d\} \Rightarrow {}^{0.5}/e$ E: no, ex. Uni Erlangen	Q: $\{^{0.5}/b, {}^{0.5}/d\} \Rightarrow {}^{0.5}/e$ E: no, ex. Uni Erlangen
6	Q: $\{^{0.5}/d, e\} \Rightarrow {}^{0.5}/c$ E: yes	Q: $\{^{0.5}/d, e\} \Rightarrow {}^{0.5}/c$ E: yes
7	Q: $\{^{0.5}/a, {}^{0.5}/b, {}^{0.5}/c, {}^{0.5}/d, e\} \Rightarrow b$ E: no, ex. Uni Saarbrücken	Q: $\{^{0.5}/a, {}^{0.5}/b, {}^{0.5}/c, {}^{0.5}/d, e\} \Rightarrow b$ E: no, ex. Uni Saarbrücken
8	Q: $\{^{0.5}/a, {}^{0.5}/b, {}^{0.5}/c, d, e\} \Rightarrow \{a, c\}$ E: no, ex. Uni Heidelberg	Q: $d \Rightarrow \{^{0.5}/a, {}^{0.5}/c, e\}$ E: yes
9	Q: $\{c, {}^{0.5}/d\} \Rightarrow \{a, {}^{0.5}/b, d, {}^{0.5}/e\}$ E: yes	Q: $\{^{0.5}/a, {}^{0.5}/b, {}^{0.5}/c, d, e\} \Rightarrow \{a, c\}$ E: no, ex. Uni Heidelberg
10	Q: $\{b, {}^{0.5}/d\} \Rightarrow \{^{0.5}/a, {}^{0.5}/c\}$ E: yes **exploration stopped.**	Q: $\{c, {}^{0.5}/d\} \Rightarrow \{a, {}^{0.5}/b, d, {}^{0.5}/e\}$ E: yes
11		Q: $\{b, {}^{0.5}/d\} \Rightarrow \{^{0.5}/a, {}^{0.5}/c, {}^{0.5}/e\}$ E: yes
12		Q: $\{a, {}^{0.5}/d\} \Rightarrow \{^{0.5}/b, {}^{0.5}/c, {}^{0.5}/e\}$ E: yes
13		Q: $\{a, {}^{0.5}/b, {}^{0.5}/c, {}^{0.5}/d, e\} \Rightarrow \{c, d\}$ E: yes
14		Q: $\{a, b^{0.5}/c, {}^{0.5}/d, {}^{0.5}/e\} \Rightarrow \{c, d, e\}$ E: yes **exploration stopped.**

Fig. 3. Exploration with and without background knowledge.

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