An interpretation of the L-Fuzzy Concept Analysis as a tool for the Morphological Image and Signal Processing

Cristina Alcalde[1], Ana Burusco[2], and Ramón Fuentes-González[2]

[1] Dpt. Matemática Aplicada. Escuela Univ. Politécnica
Univ. del País Vasco. Plaza de Europa, 1
20018 - San Sebastián (Spain)
c.alcalde@ehu.es
[2] Dpt. Automática y Computación. Univ. Pública de Navarra
Campus de Arrosadía
31006 - Pamplona (Spain)
{burusco,rfuentes}@unavarra.es

Abstract. In this work we are going to set up a new relationship between the L-fuzzy Concept Analysis and the Fuzzy Mathematical Morphology. Specifically we prove that the problem of finding fuzzy images or signals that remain invariant under a fuzzy morphological opening or under a fuzzy morphological closing, is equal to the problem of finding the L-fuzzy concepts of some L-fuzzy context. Moreover, since the Formal Concept Analysis and the Mathematical Morphology are the particular cases of the fuzzy ones, the showed result has also an interpretation for binary images or signals.

Keywords: L-fuzzy Concept Analysis, Fuzzy Mathematical Morphology, Morphological Image Processing

1 Introduction

The L-fuzzy Concept Analysis and the Fuzzy Mathematical Morphology were developed in different contexts but both use the lattice theory as algebraic framework.

In the case of the L-fuzzy Concept Analysis, we define the L-fuzzy concepts using a fuzzy implication and a composition operator associated with it. In the Fuzzy Mathematical Morphology, a fuzzy implication is also used to define the erosion but a t-norm also appears to introduce the dilation.

On the other hand, both theories have been used in knowledge extraction processes in data bases [14–16].

Next, we will show a brief description of them.

ISBN 978–84–695–5252–0, Universidad de Málaga (Dept. Matemática Aplicada)

2 Antecedents

2.1 *L*-fuzzy Concept Analysis

The Formal Concept Analysis of R. Wille [28, 17] extracts information from a binary table that represents a formal context (X, Y, R) with X and Y finite sets of objects and attributes respectively and $R \subseteq X \times Y$. The hidden information is obtained by means of the formal concepts that are pairs (A, B) with $A \subseteq X$, $B \subseteq Y$ verifying $A^* = B$ and $B^* = A$, where $*$ is the derivation operator that associates the attributes related to the elements of A to every object set A, and the objects related to the attributes of B to every attribute set B. These formal concepts can be interpreted as a group of objects A that shares the attributes of B.

In previous works [11, 12] we have defined the L-fuzzy contexts (L, X, Y, R), with L a complete lattice, X and Y sets of objects and attributes respectively and $R \in L^{X\times Y}$ a fuzzy relation between the objects and the attributes. This is an extension of the Wille's formal contexts to the fuzzy case when we want to study the relationship between the objects and the attributes with values in a complete lattice L, instead of binary values.

In our case, to work with these L-fuzzy contexts, we have defined the derivation operators 1 and 2 given by means of these expressions:

$$\forall A \in L^X, \forall B \in L^Y \quad A_1(y) = \inf_{x\in X}\{I(A(x), R(x, y))\}$$
$$B_2(x) = \inf_{y\in Y}\{I(B(y), R(x, y))\}$$

with I a fuzzy implication operator defined in the lattice $(L, \leq)$ and where A_1 represents the attributes related to the objects of A in a fuzzy way, and B_2, the objects related to all the attributes of B.

In this work, we are going to use the following notation for these derivation operators to stand out their dependence to relation R:

$\forall A \in L^X$, $\forall B \in L^Y$, we define $\mathcal{D}_R : L^X \to L^Y$, $\mathcal{D}_{R^{op}} : L^Y \to L^X$

$$\mathcal{D}_R(A)(y) = A_1(y) = \inf_{x\in X}\{I(A(x), R(x, y))\}$$
$$\mathcal{D}_{R^{op}}(B)(x) = B_2(x) = \inf_{y\in Y}\{I(B(y), R^{op}(y, x))\}$$

where we denote by R^{op} the opposite relation of R, that is, $\forall (x, y) \in X \times Y$, $R^{op}(y, x) = R(x, y)$.

The information stored in the context is visualized by means of the L-fuzzy concepts that are some pairs $(A, A_1) \in (L^X, L^Y)$ with $A \in \text{fix}(\varphi)$, set of fixed points of the operator φ, being defined from the derivation operators 1 and 2 as $\varphi(A) = (A_1)_2 = A_{12}$. These pairs, whose first and second components are said to be the fuzzy extension and intension respectively, represent a set of objects that share a set of attributes in a fuzzy way.

The set $\mathcal{L} = \{(A, A_1)/A \in \text{fix}(\varphi)\}$ with the order relation $\leq$ defined as:

$$\forall (A, A_1), (C, C_1) \in \mathcal{L}, \quad (A, A_1) \leq (C, C_1) \text{ if } A \leq C (\text{ or } A_1 \geq C_1)$$

is a complete lattice that is said to be the L-fuzzy concept lattice [11, 12].

On the other hand, given $A \in L^X$, (or $B \in L^Y$) we can obtain the associated L-fuzzy concept. In the case of using a residuated implication, as we do in this work, the associated L-fuzzy concept is (A_{12}, A_1) (or (B_2, B_{21})).

Other important results about this theory are in [1, 10, 25, 13, 24, 5].

A very interesting particular case of L-fuzzy contexts appears trying to analyze situations where the objects and the attribute sets are coincident [2, 3], that is, L-fuzzy contexts (L, X, X, R) with $R \in L^{X \times X}$, (this relation can be reflexive, symmetrical ...). In these situations, the L-fuzzy concepts are pairs (A, B) such that $A, B \in L^X$.

These are the L-fuzzy contexts that we are going to use to obtain the main results of this work. Specifically, we are going to take a complete chain $(L, \leq)$ as the valuation set, and L-fuzzy contexts as $(L, \mathbb{R}^n, \mathbb{R}^n, R)$ or $(L, \mathbb{Z}^n, \mathbb{Z}^n, R)$. In the first case, the L-fuzzy concepts (A, B) are interpreted as signal or image pairs related by means of R. In the second case, A and B are digital versions of these signals or images.

2.2 Mathematical Morphology

The Mathematical Morphology is a theory concerned with the processing and analysis of images or signals using filters and operators that modify them. The fundamentals of this theory (initiated by G. Matheron [22, 23] and J. Serra [26]), are in the set theory, the integral geometry and the lattice algebra. Actually this methodology is used in general contexts related to activities as the information extraction in digital images, the noise elimination or the pattern recognition.

Mathematical Morphology in binary images and grey levels images In this theory images A from $X = \mathbb{R}^n$ or $X = \mathbb{Z}^n$ (digital images or signals when n=1) are analyzed.

The *morphological filters* are defined as operators $F : \wp(X) \to \wp(X)$ that transform, symplify, clean or extract relevant information from these images $A \subseteq X$, information that is encapsulated by the filtered image $F(A) \subseteq X$.

These morphological filters are obtained by means of two basic operators, the *dilation* δ_S and the *erosion* ε_S, that are defined in the case of binary images with the *sum* and *difference of Minkowski* [26], respectively.

$$\delta_S(A) = A \oplus S = \bigcup_{s \in S} A_s \quad \varepsilon_S(A) = A \ominus \check{S} = \bigcap_{s \in \check{S}} A_s$$

where A is an image that is treated with another $S \subseteq X$, that is said to be *structuring element*, or with its opposite $\check{S} = \{-x / x \in S\}$ and where A_s represents a translation of A: $A_s = \{a + s / a \in A\}$.

The structuring image S represents the effect that we want to produce over the initial image A.

These operators are not independent since they are dual transformations with respect to the complementation [27], that is, if A^c represents the complementary set of A, then:

$$\varepsilon_S(A) = (\delta_S(A^c))^c, \forall A, S \in \wp(X)$$

We can compose these operators dilation and erosion associated with the structuring element S and obtain the basic filters *morphological opening* $\gamma_S : \wp(X) \to \wp(X)$ and *morphological closing* $\phi_S : \wp(X) \to \wp(X)$ defined by:

$$\gamma_S = \delta_S \circ \varepsilon_S \quad \phi_S = \varepsilon_S \circ \delta_S$$

The opening γ_S and the closing ϕ_S over these binary images verifies the two conditions that characterize the morphological filters: They are isotone and idempotent operators, and moreover it is verified, $\forall A, S \in \wp(X)$:

a) $\gamma_S(A) \subseteq A \subseteq \phi_S(A)$

b) $\gamma_S(A) = (\phi_S(A^c))^c$

These operators will characterize some special images (the S-open and the S-closed ones) that will play an important role in this work.

This theory is generalized introducing some tools to treat images with grey levels [26]. The images and the structuring elements are now maps defined in $X = \mathbb{R}^n$ and with values in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ or defined in $X = \mathbb{Z}^n$ and with values in finite chains as, for instance, $\{0, 1, \dots, 255\}$.

The previous definitions can be immersed in a more general framework that considers each image as a point $x \in L$ of a partially ordered structure $(L, \leq)$ (complete lattice), and the filters as operators $F : L \to L$ with properties related to the order in these lattices [26, 19].

Now, the erosions $\varepsilon : L \to L$ are operators that preserve the infimum $\varepsilon(\inf M) = \inf \varepsilon(M), \forall M \subseteq L$ and the dilations $\delta : L \to L$, the supremum: $\delta(\sup M) = \sup \delta(M), \forall M \subseteq L$. The opening $\gamma : L \longrightarrow L$ and the closing $\phi : L \longrightarrow L$ are isotone and idempotent operators verifying $\gamma(A) \leq A \leq \phi(A)$.

Fuzzy Mathematical Morphology In this new framework and associated with lattices, a new *fuzzy morphological image processing* has been developed [6, 7, 4, 8, 9, 21, 20] using L-fuzzy sets A and S (with $X = \mathbb{R}^2$ or $X = \mathbb{Z}^2$) as images and structuring elements.

In this interpretation, the filters are operators $F_S : L^X \to L^X$, where L is the chain $[0, 1]$ or a finite chain $L_n = \{0 = \alpha_1, \alpha_2, ..., \alpha_{k-1}, 1\}$ with $0 < \alpha_1 < ... < \alpha_{k-1} < 1$.

In all these cases, *fuzzy morphological dilations* $\delta_S : L^X \to L^X$ and *fuzzy morphological erosions* $\varepsilon_S : L^X \to L^X$ are defined using some operators of the fuzzy logic [4, 6, 9, 21].

In general, there are two types of relevant operators in the Fuzzy Mathematical Morphology. One of them is formed by those obtained by using some pairs $(*, I)$ of adjunct operators related by:

$$(\alpha * \beta \leq \psi) \Longleftrightarrow (\beta \leq I(\alpha, \psi))$$

The other type are the morphological operators obtained by pairs $(*, I)$ related by a strong negation $' : L \to L$:

$$\alpha * \beta = (I(\alpha, \beta'))', \forall (\alpha, \beta) \in L \times L$$

An example of one of these pairs that belongs to both types is the formed by the t-norm and the implication of Lukasiewicz.

In this paper, we work taking as $(X, +)$ the commutative group $(\mathbb{R}^n, +)$ or the commutative group $(\mathbb{Z}^n, +)$, and as $(L, \leq, ', I, *)$, the complete chain $L = [0, 1]$ or a finite chain as $L = L_n = \{0 = \alpha_1, \alpha_2, ..., \alpha_{k-1}, 1\}$ with the Zadeh negation and $(*, I)$ the Lukasiewicz t-norm and implication.

We interpret the L-fuzzy sets $A : X \to L$ and $S : X \to L$ as n-dimensional images in the space $X = \mathbb{R}^n$ (or n-dimensional digital images in the case of $X = \mathbb{Z}^n$).

In the literature, (see [4, 6, 18, 21]), erosion and dilation operators are introduced associated with the residuated pair $(*, I)$ as follows:

If $S : X \to L$ is an image that we take as *structuring element*, then we consider the following definitions associated with (L, X, S)

Definition 1. *[6] The fuzzy erosion of the image $A \in L^X$ by the structuring element S is the L-fuzzy set $\varepsilon_S(A) \in L^X$ defined as:*

$$\varepsilon_S(A)(x) = \inf\{I(S(y - x), A(y))/y \in X\} \qquad \forall x \in X$$

The fuzzy dilation of the image A by the structuring element S is the L-fuzzy set $\delta_S(A)$ defined as:

$$\delta_S(A)(x) = \sup\{S(x - y) * A(y)/y \in X\} \qquad \forall x \in X$$

Then we obtain fuzzy erosion and dilation operators $\varepsilon_S, \delta_S : L^X \to L^X$. Moreover, it is verified:

Proposition 1. *(1) If $\leq$ represents now the usual order in L^X obtained by the order extension in the chain L, then the pair $(\varepsilon_S, \delta_S)$ is an adjunction in the lattice $(L^X, \leq)$, that is:*

$$\delta_S(A_1) \leq A_2 \Longleftrightarrow A_1 \leq \varepsilon_S(A_2)$$

(2) If A' is the negation of A defined by $A'(x) = (A(x))', \forall x \in X$ and if $\breve{S}$ represents the image associated with S such that $\breve{S}(x) = S(-x), \forall x \in X$, then it is verified:

$$\varepsilon_S(A') = (\delta_{\breve{S}}(A))', \quad \delta_S(A') = (\varepsilon_{\breve{S}}(A))', \qquad \forall A, S \in L^X$$

Proof. (1) Suppose that $\delta_S(A_1) \leq A_2$. Then $\delta_S(A_1)(x) \leq A_2(x)\ \forall x \in X$. That is, $S(x-y) * A_1(y) \leq A_2(x)\ \forall (x,y) \in X \times X$.

From these inequalities and from the equivalence $\alpha * \beta \leq \gamma \Leftrightarrow \beta \leq I(\alpha, \gamma)$:

$$A_1(y) \leq I(S(x-y), A_2(x)), \forall (x,y) \in X \times X$$

and interchanging x and y, we have:

$$A_1(x) \leq I(S(y-x), A_2(y)), \forall (x,y) \in X \times X$$

and consequently

$$A_1(x) \leq \inf\{I(S(y-x), A_2(y))/y \in X\}, \forall x \in X$$

That is: $A_1(x) \leq \varepsilon_S(A_2)(x), \forall x \in X$ that shows that $A_1 \leq \varepsilon_S(A_2)$.

We can prove the other implication in a similar way.

(2) Let be $x \in X$.

$$\begin{aligned}\varepsilon_S(A')(x) &= \inf\{I(S(y-x), A'(y))/y \in X\} = \inf\{I(\check{S}(x-y), A'(y))/y \in X\} \\ &= \inf\{(\check{S}(x-y) * A(y))'/y \in X\} = (\sup\{(\check{S}(x-y) * (A(y))/y \in X\})' \\ &= (\delta_{\check{S}}(A)(x))' = (\delta_{\check{S}}(A))'(x)\end{aligned}$$

The second equality is proved analogously. □

3 Relation between both theories

The erosion and dilation operators given in Definition 1 are used to construct the basic morphological filters: the opening and the closing (see [4, 6, 18, 21]).

Definition 2. *The fuzzy opening of the image $A \in L^X$ by the structuring element $S \in L^X$ is the fuzzy subset $\gamma_S(A)$ that results from the composition of the erosion $\varepsilon_S(A)$ of A by S followed by its dilation:*

$$\gamma_S(A) = \delta_S(\varepsilon_S(A)) = (\delta_S \circ \varepsilon_S)(A)$$

The fuzzy closing of the image $A \in L^X$ by the structuring element $S \in L^X$ is the fuzzy subset $\phi_S(A)$ that results from the composition of the dilation $\delta_S(A)$ of A by S followed by its erosion:

$$\phi_S(A) = \varepsilon_S(\delta_S(A)) = (\varepsilon_S \circ \delta_S)(A)$$

It can be proved that the operators γ_S and ϕ_S are morphological filters, that is, they preserve the order and they are idempotent:

$$A_1 \leq A_2 \Longrightarrow (\gamma_S(A_1) \leq \gamma_S(A_2)) \text{ and } (\phi_S(A_1) \leq \phi_S(A_2))$$

$$\gamma_S(\gamma_S(A)) = \gamma_S(A), \phi_S(\phi_S(A)) = \phi_S(A), \forall A \in L^X, \forall S \in L^X$$

Moreover, these filters verify that:

$$\gamma_S(A) \leq A \leq \phi_S(A) \quad \forall A \in L^X, \forall S \in L^X$$

Analogous results that those obtained for the erosion and dilation operators can be proved for the opening and closing:

Proposition 2. *If A' is the negation of A defined by $A'(x) = (A(x))' \quad \forall x \in X$, then:*

$$\gamma_S(A') = (\phi_{\breve{S}}(A))', \quad \phi_S(A') = (\gamma_{\breve{S}}(A))' \qquad \forall A, S \in L^X$$

Proof. $\gamma_S(A') = \delta_S(\varepsilon_S(A')) = \delta_S((\delta_{\breve{S}}(A))') = (\varepsilon_{\breve{S}}(\delta_{\breve{S}}(A)))' = (\phi_{\breve{S}}(A))'$.

The other equality can be proved in an analogous way. □

Since the operators γ_S and ϕ_S are increasing in the complete lattice $(L^X, \leq)$, by Tarski's theorem, the respective fixed points sets are not empty. These fixed points will be used in the following definition:

Definition 3. *An image $A \in L^X$ is said to be S-open if $\gamma_S(A) = A$ and it is said to be S-closed if $\phi_S(A) = A$.*

These S-open and S-closed sets provide a connection between the Fuzzy Mathematical Morphology and the Fuzzy Concept Theory, as we will see next.

For that purpose, given the complete chain L that we are using, and a commutative group $(X, +)$, we will associate with any fuzzy image $S \in L^X$, the fuzzy relation $R_S \in L^{X \times X}$ such that:

$$R_S(x, y) = S(x - y), \forall (x, y) \in X \times X$$

It is evident that $R_{S'} = R_S^{'}$ and, if R_S^{op} represents the opposite relation of R_S, then $R_S^{op} = R_{\breve{S}}$.

In agreement with this last point, we can redefine the erosion and dilation as follows:

$$\begin{aligned}
\varepsilon_S(A)(x) &= \inf\{I(R_S(y, x), A(y))/y \in X\} \\
&= \inf\{I(R_S^{op}(x, y), A(y))/y \in X\}, \forall x \in X \\
\delta_S(A)(x) &= \sup\{R_S(x, y) * A(y)/y \in X\}, \forall x \in X
\end{aligned}$$

With this rewriting, given the structuring element $S \in L^X$, we can interpret the triple (L, X, S) as an L-fuzzy context (L, X, X, R'_S) where the sets of objects and attributes are coincident. The incidence relation $R'_S \in L^{X \times X}$ is at the same time the negation of an interpretation of the fuzzy image by the structuring element S.

We will use this representation as L-fuzzy context to prove the most important results that connect both theories:

Theorem 1. *Let (L, X, S) be the triple associated with the structuring element $S \in L^X$. Let $(L, X, X, R_S^{'})$ be the L-fuzzy context whose incidence relation $R_S^{'} \in L^{X \times X}$ is the negation of the relation R_S associated with S. Then the operators erosion ε_S and dilation δ_S en (L, X, S) are related to the derivation operators $\mathcal{D}_{R'_S}$ and $\mathcal{D}_{R_S'^{op}}$ in the L-Fuzzy context $(L, X, X, R_S^{'})$ by:*

$$\varepsilon_S(A) = \mathcal{D}_{R'_S}(A') \quad \forall A \in L^X$$
$$\delta_S(A) = (\mathcal{D}_{R_S'^{op}}(A))' \quad \forall A \in L^X$$

Proof. Taking into account the properties of the Lukasiewicz implication, for any $x \in X$, it is verified that:

$\varepsilon_S(A)(x) = \inf\{I(R_S(y, x), A(y))/y \in X\} = \inf\{I(A'(y), R_S^{'}(y, x))/y \in X\} = \mathcal{D}_{R'_S}(A')(x)$

Analogously,

$\delta_S(A)(x) = \sup\{R_S(x, y) * A(y)/y \in X\} = \sup\{(I(R_S(x, y), A'(y)))'/y \in X\} =$
$(\inf\{I(R_S(x, y), A'(y))/y \in X\})' = (\inf\{I(A(y), R_S^{'}(x, y))/y \in X\})' =$
$(\inf\{I(A(y), R_S^{'op}(y, x))/y \in X\})' = ((\mathcal{D}_{R_S'^{op}}(A))(x))' = (\mathcal{D}_{R_S'^{op}}(A))'(x)$. □

As a consequence, we obtain the following result which proves the connection between the outstanding morphological elements and the L-fuzzy concepts:

Theorem 2. *Let be $S \in L^X$ and let be $R_S \in L^{X \times X}$ its associated relation. The following propositions are equivalent:*

1. *The pair $(A, B) \in L^X \times L^X$ is an L-fuzzy concept of the context $(L, X, X, R_S^{'})$, where $R_S^{'}(x, y) = S'(x - y)$ $\forall (x, y) \in X \times X$.*
2. *The pair $(A, B) \in L^X \times L^X$ is such that the negation A' of A is S-open ($\gamma_S(A') = A'$) and B is the S-erosion of A' (that is, $B = \varepsilon_S(A')$).*
3. *The pair $(A, B) \in L^X \times L^X$ is such that B is S-closed ($\phi_S(B) = B$) and A is the negation of the S-dilation of B (that is, $A = (\delta_S(B))'$).*

Proof.

1 $\Longrightarrow$ 2) Let be $S \in L^X$ and $R_S \in L^{X \times X}$ its associated relation. Let us consider an L-fuzzy concept (A, B) of the L-fuzzy context $(L, X, X, R_S^{'})$ in which $R_S^{'}$ is the negation of R_S. Then, it is verified that $B = \mathcal{D}_{R'_S}(A)$ and $A = \mathcal{D}_{R_S'^{op}}(B)$, and, by the previous proposition, $\varepsilon_S(A') = \mathcal{D}_{R'_S}(A) = B$. Moreover, it is fulfilled that $\gamma_S(A') = \delta_S(\varepsilon_S(A')) = \delta_S(B) = (\mathcal{D}_{R_S'^{op}}(B))' = A'$ which proves that A' is S-open.

2 $\Longrightarrow$ 3) Let us suppose that the hypothesis of *2* are fulfilled. Then, $\phi_S(B) = \varepsilon_S(\delta_S(B)) = \varepsilon_S(\delta_S(\varepsilon_S(A'))) = \varepsilon_S(\gamma_S(A')) = \varepsilon_S(A') = B$, which proves that B is S-closed. On the other hand, from the hypothesis $B = \varepsilon_S(A')$ can be deduced that $\delta_S(B) = \delta_S(\varepsilon_S(A')) = \gamma_S(A')$, and consequently, taking into account that A' is S-open, that $\delta_S(B) = A'$, and finally, $A = (\delta_S(B))'$.

3 $\Longrightarrow$ 1) Let (A, B) be a pair fulfilling the hypothesis of *3*. Let us consider the L-fuzzy context (L, X, X, R'_S). Then, by the previous theorem we can deduce that $(\mathcal{D}_{R'^{op}_S}(B)) = (\delta_S(B))' = A$. On the other hand, applying the previous theorem and the hypothesis, $\mathcal{D}_{R'_S}(A) = \varepsilon_S(A') = \varepsilon_S(\delta_S(B)) = \phi_S(B) = B$, which finishes the proof. □

Let us see now some examples.

Example 1. Interpretation of some binary images as formal concepts.

In the referential set $X = \mathbb{R}^2$, if $\overline{\boldsymbol{x}} = (x_1, x_2) \in \mathbb{R}^2$, w is a positive number and if S is the structuring binary image

$$S = \{(x_1, x_2) \in \mathbb{R}^2 / x_1^2 + x_2^2 \leq w^2\}$$

then the associated incidence relation $R_S^c \subset \mathbb{R}^2 \times \mathbb{R}^2$ is such that:

$$\overline{\boldsymbol{x}} R_S^c \overline{\boldsymbol{y}} \Longleftrightarrow ((x_1 - y_1)^2 + (x_2 - y_2)^2 > w^2)$$

which is irreflexive and transitive. The pair (A, B) showed in Figure 1 is a concept of the context $(\mathbb{R}^2, \mathbb{R}^2, R_S^c)$, because $\gamma_S(A^c) = A^c$ and $B = \varepsilon_S(A^c)$.

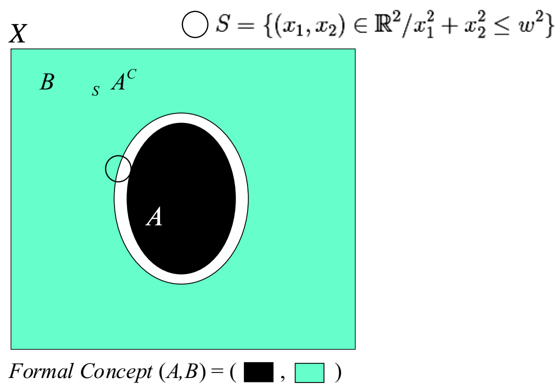


Fig. 1. A formal concept of the context $(\mathbb{R}^2, \mathbb{R}^2, R_S^c)$.

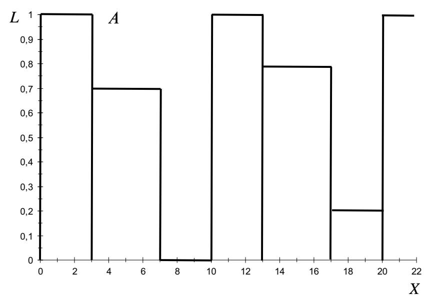
Example 2. Interpretation of some open digital signals as fuzzy concepts.

It is known that the erosion $\varepsilon_S(A)$ of an image A by a binary structuring element S can be rewritten in terms of infimum of the traslations of A by elements of S [27]:

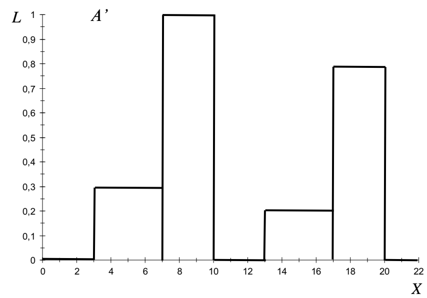
$$\varepsilon_S(A) = \bigwedge_{s \in S} A_{-s} \quad \text{where } A_k(x) = A(x - k)$$

If $X \subseteq \mathbb{Z}$ and $L = \{0, 0.1, 0.2, ..., 0.9, 1\}$ then, the maps $A : X \to L$ can be interpret as 1-D discrete signals. In Figure 2 there are some examples of discrete signals.

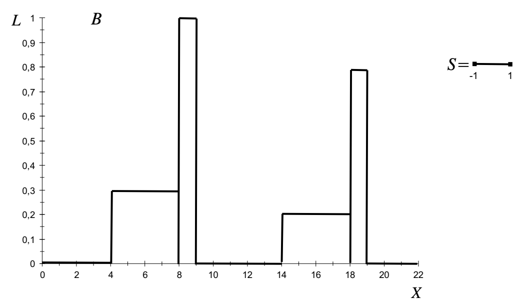
The signal in Fig 2(c) is the erosion of A' in Fig 2(b), using a line segment of three pixels as a structuring element, the middle pixel being its origin (S is a crisp set).



(a) Discrete signal A as an L-Fuzzy set



(b) Negation A' of the discrete signal A



(c) Discrete signal $B = \varepsilon_S(A')$

Fig. 2. Discrete signals

Here, we can also find the erosions in terms of intersections of image traslations: $\varepsilon_S(A') = \bigwedge\{A'_{-1}, A'_0, A'_{+1}\}$.

It can be proved that A' verifies $\gamma_S(A') = A'$. So, with A in Fig 2 the pair (A, B) with $B = \varepsilon_S(A')$ is a fuzzy concept of the context $(L, \mathbb{Z}, \mathbb{Z}, R^c_S)$ with the crisp incidence relation $xR^c_S y \Leftrightarrow (x - y) \notin S$, (that is, $|x - y| > 1$).

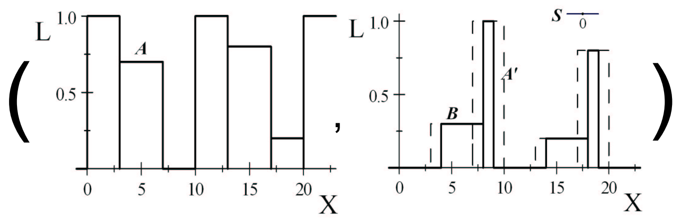


Fig. 3. L-Fuzzy concept (A, B) of the L-Fuzzy context $(L, \mathbb{Z}, \mathbb{Z}, R_S^c)$

4 Conclusions and Future work

The main results of this work show an interesting relation between the L-fuzzy Concept Analysis and the Fuzzy Mathematical Morphology that we want to develop in future works. So, we can apply the algorithms for the calculus of L-fuzzy concepts in Fuzzy Mathematical Morphology and vice versa.

On the other hand, we are extending these results to other type of operators as other implications, t-norms, conjunctive uninorms etc... and to some L-fuzzy contexts where the objects and the attributes are not related to signal or images.

Acknowledgements

This work has been partially supported by the Research Group "Intelligent Systems and Energy (SI+E)" of the Basque Government, under Grant IT519-10, and by the Research Group "Adquisición de conocimiento y minería de datos, funciones especiales y métodos numéricos avanzados" of the Public University of Navarra.

References

1. Alcalde, C., Burusco, A., Fuentes-González, R., Zubia, I.: Treatment of L-fuzzy contexts with absent values. Information Sciences. 179 (1-2), 1–15 (2009)
2. Alcalde, C., Burusco, A., Fuentes-González, R.: Contextos con conjuntos de objetos y atributos iguales. In: Proc. of ESTYLF08, pp. 85–89. Mieres - Langreo (2008)
3. Alcalde, C., Burusco, A., Fuentes-González, R.: Analysis of some L-fuzzy relational equations and the study of its solutions by means of the L-fuzzy Concept Theory. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems. 20 (1), 21–40 (2012)
4. De Baets, B.: Fuzzy morphology: a logical approach. In: Ayyub, B., Gupta, M. (eds.) Uncertainty Analysis, Engineering and Science: Fuzzy Logic, Statistics and neural network Approach, pp. 53–67, Kluwer Academic Publishers (1997)
5. Belohlavek, R., Konecny, J.: Concept lattices of isotone vs. antitone Galois connections in graded setting: Mutual reducibility revisited. Information Sciences. 199, 133–137 (2012)

6. Bloch, I., Maître, H.: Fuzzy Mathematical Morphologies: a comparative study. Télécom Paris 94D001 (1994)
7. Bloch, I.: Duality vs.adjunction for fuzzy mathematical morphology and general form of fuzzy erosions and dilations. Fuzzy Sets and Systems. 160, 1858–1867 (2009)
8. Burillo, P., Fuentes-González, R., Frago, N.: Inclusion grade and fuzzy implication operators. Fuzzy Sets and Systems. 114, 417–429 (2000)
9. Burillo, P., Frago, N., Fuentes-González, R.: Generation of Fuzzy Mathematical Morphologies. Mathware & Soft Computing. VIII (1), 31–46 (2001)
10. Burusco, A., Fuentes-González, R.: Concept lattices defined from implication operators. Fuzzy Sets and Systems. 114, 431–436 (2000)
11. Burusco, A., Fuentes-González, R.: The Study of the L-fuzzy Concept Lattice. Mathware and Soft Computing. I (3), 209–218 (1994)
12. Burusco, A., Fuentes-González, R.: Construction of the L-fuzzy Concept Lattice. Fuzzy Sets and Systems. 97 (1), 109–114 (1998)
13. Djouadi, Y., Prade, H.: Interval-Valued Fuzzy Galois Connections: Algebraic Requirements and Concept Lattice Construction. Fundamenta Informaticae. 99 (2), 169–186 (2010)
14. Frago, N., Fuentes-González, R.: Descubrimiento de Conocimiento en Bases de Datos Utilizando Técnicas de Morfología Matemática Borrosa. Información tecnológica. 18 (6), 39–50 (2007)
15. Frago, N., Fuentes-González, R.: Procesos de descubrimiento de conocimiento en bases de datos usando grafos asociados a filtros morfológicos. In: Proc. of ESTYLF08, pp. 85–89. Mieres - Langreo (2008)
16. Fuentes-González, R.: The incorporation of the Mathematical Morphology, the Formal Concept Analysis and the Fuzzy Logic techniques and tools to the data cleaning, aggregating, reducing and mining stages of the Knowledge Discovery process. In: Troya, J.M., Ossowski, S. (eds.) Proceedings of CEDI, pp. 147–154. Granada (2005)
17. Ganter, B., Wille, R.: Formal concept analysis: Mathematical foundations. Springer, Berlin - New York (1999)
18. Goutsias, J., Heijmans, H.J.A.M.: Fundamenta Morphologicae Mathematicae. Fundamenta Informaticae. 41, 1–31 (2000)
19. Heijmans, H.J.A.M.: Morphological Image Operators. Academic Press Inc. (1994)
20. Maragos, P.: Lattice Image Precessing: A Unification of Morphological and Fuzzy Algebric Systems. Journal of Mathematical Imaging and Vision. 22, 333–353 (2005)
21. Mas, M., Monserrat, M., Torrens, J.: S-implications and R-implications on a finite chain. Kybernetika. 40 (1), 3–20 (2004)
22. Matheron, G.: Eléments pour une théorie des milieux poreux. Masson, Paris (1967)
23. Matheron, G.: Random Sets and Integral Geometry. Wiley, New York (1975)
24. Medina, J.: Multi-adjoint property-oriented concept lattices. Information Sciences. 190, 95–106 (2012)
25. Medina, J., Ojeda-Aciego, M.: Multi-adjoint t-concept lattices. Information Sciences. 180 (5), 712–725 (2010)
26. Serra, J.: Image Analysis and Mathematical Morphology. Academic Press. I (fourth printing 1990) and II (second printing 1992)
27. Soille, P.: Morphological Image Analysis. Principles and Applications. Second Edition. Springer (2004)
28. Wille, R.: Restructuring lattice theory: an approach based on hierarchies of concepts. In: Rival, I. (ed.), Ordered Sets, pp. 445–470. Reidel, Dordrecht (1982)