Fuzzy formal concept analysis via multilattices: first prospects and results*

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Abstract. The most general algebraic structure of truth-values considered in the theory of fuzzy concept analysis to evaluate the attributes and objects has been a lattice. However, in some examples arises the necessity of a more general structure. In this paper we investigate the use of multilattices as underlying set of truth-values for these attributes and objects.

1 Introduction

The study of reasoning methods to work out with uncertainty, imprecise data or incomplete information has been a trending topic in the recent years in order to explain, in a better way, observed facts, specify statements, reasoning and/or execute programs.

One important and powerful mathematical tool that has been used for this purpose at theoretical level is fuzzy logic. From the applicative side, neural networks have a massively parallel architecture-based dynamics which are inspired by the structure of human brain, adaptation capabilities, and fault tolerance. The recent paradigm of soft computing promotes the use and integration of different approaches for the problem solving.

Formal concept analysis, introduced by Wille in [23], is a useful tool for qualitative data analysis and has become an appealing major research topic, from both the theoretical and applied perspectives. What we pretend in this paper is to present the multilattices as a basis on the area of formal concept analysis and, specifically, what will lead us to what we have called fuzzy formal concept multilattice.

There has been many approaches in order to generalise the classical concept lattices given by Ganter and Wille [12] allowing some uncertainty in data. The first *fuzzy* approach was proposed by Burusco and Fuentes-González [4] where fuzzy concept lattices were first presented, and later further developed by Pollandt [22] and Bělohlávek [1]. Other approaches emerge trying to work with non-commutative fuzzy logic and similarity as we can see in the work of

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Georgescu and Popescu [13]. This approach, consisting in generalizing the equality relation and considering an alternative similarity relation, underlies in the works of Bělohlávek [2], which considered L-equalities to extend the fuzzy concept lattice. This approach was extended in an asymmetric way, although only for the case of classical equality ($L = \{0, 1\}$) by Krajči, who introduced the so-called generalized concepts lattices in [16,15].

Recently, a new approach has been proposed by Medina et al in [20,19] who introduced the multi-adjoint concept lattices, joining the multi-adjoint philosophy with concept lattices. To do this the authors needed to generalize the adjoint pairs into what they called adjoint triples [6]. This new structure directly generalizes almost all the approaches previously cited.

On the other hand, the theory of multilattices arose trying to weaken the restrictions imposed on a (complete) lattice, namely, the "existence of least upper bounds and greatest lower bounds" is relaxed to "existence of *minimal* upper bounds and *maximal* lower bounds". In this direction, several definitions have been proposed in the mathematical literature of the structure so-called multilattice [3,14].

Later on, an alternative notion of multilattice, with better properties regarding substructures than the previous definitions, has been introduced [5,17]. Moreover, this structure has proved to be an important tool in order to obtain some advances in the theory of mechanized deduction in temporal logics.

Multilattices, in the sense of the paragraph above, also arise in a natural manner in the research area concerning fuzzy extensions of logic programming [18]. For instance, one of the hypotheses of the main termination result for sorted multi-adjoint logic programs [7] has been weakened only when the underlying set of truth-values is a multilattice [8]; as far as we know, the question of providing a counter-example on a lattice remains open.

The main result introduced here is the presentation of multilattices as underlying set where to evaluate the attributes, the objects and the relation in a fuzzy environment to formal concept analysis, indeed, this leads us to see that the set of "multilattice concepts" is a multilattice. We can see as well that if we evaluate the objects or the attributes in a lattice and the other in a multilattice what we have again a lattice not a multilattice as we could think at first.

The idea of using multilattices as underlying set of truth values arises us since in real life many thigns are ordered in a way that we we know that some objects are better than others but sometimes we can not choose which is the best of them because they can have different properties. This idea can be seen in the example we give in the last section of this paper where we consider a group of hotels. It is logical to think that four-star hotels are better than three star hotel, but sometimes we can not decide if a four-star hotel is better than another four-star hotel since these can have different properties to be considered like acommodation, location, price, etc., and one can be better in one property and worse in other. This, as we have already said, is what have lead up to consider multilattices because this structure deals better with objects which are uncomparable.

The plan of this paper is the following: in Section 2 we present the main definitions and results to understand the paper. Section 3 presents the formal concept multilattice; we also provide an example where this new structure can be used in Section 4; the paper ends with some conclusions and prospects for future work.

2 Preliminaries

In this first section we will set the basic notions required to the complete understanding of the paper. We will start with a bit of lattice and multilattice theory and finished with concept lattice analysis.

One of the most structures used when hanging with fuzziness is a lattice, this has been a very suitable in order to develop many theories. Although the following definitions are well known, we will recall them here in order to make this paper as selfcontained as possible. The definition of a lattice is given bellow.

Definition 1. *By a* complete lattice *is understood a poset,* $(L, \preceq)$*, where every subset of* L *has supremum and infimum.*

When instead of the existence of both supremum and infimum for every subset we only ask for the existence of one of them the notion of semilattice arised.

Definition 2. *By a* complete lower semilattice *is understood a poset,* $(L, \preceq)$*, where every non-empty subset of* L *has infimum.*

Definition 3. *By a* complete upper semilattice *is a poset,* $(L, \preceq)$*, where every non-empty subset of* L *has supremum.*

Nevertheless, there is a closed relationship between lattices and semilattices, indeed, if we have the existence of a top element in lower semilattices or a bottom element in upper semilattices we have that they become lattices as the following theorem states.

Theorem 1. *A complete upper (lower) semilattice* $(L, \preceq)$ *with a minimum element (maximum element) is a complete lattice.*

Once we have reminded the notions of lattice and semilattice we will pass to define what a multilattice is. To get this we will start with some preliminaries notions.

Definition 4. *Let* $(P, \leq)$ *be a poset and* $K \subseteq P$*, we say that:*

- K *is called a* chain *if for every two elements* $x, y \in K$ *we have that* $x \leq y$ *or* $y \leq x$*.*
- K *is called an* antichain *if none of its elements are comparable, i.e., for every* $x, y \in K$ *we have that* $x \nleq y$ *and* $y \nleq x$*.*

Definition 5. *A poset* $(P, \leq)$ *is called* coherent *if every chain has supremum and infimum.*

Once we have introduced these notions we can give the definition of a complete multilattice.

Definition 6. *A* complete multilattice *is a coherent poset without infinite antichains,* $(M, \leq)$*, where for each subset, the set of its upper (lower) bounds has minimal (maximal) elements.*

Each minimal(maximal) element of the upper (lower) bounds of a subset is called *multisupremum(multinfimum)*. The set of all multisuprema(multinfima) will be denoted by multisup(multinf).

Example 1. An example of a multilattice is given in figure 1.

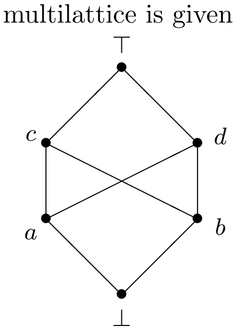

Fig. 1. Multilattice M6

In this multilattice we have that if we consider the subset $\{a, b\}$ we have that $\text{multinf}\{a, b\} = \perp$ and $\text{multisup}\{a, b\} = \{c, d\}$ while if we consider as subset $\{c, d\}$ we have that $\text{multinf}\{c, d\} = \{a, b\}$ and $\text{multisup}\{a, b\} = \top$

We will remind now the notion of adjoint pair we will use later [11,21].

Definition 7. *Let* $(P_1, \leq_1)$*,* $(P_2, \leq_2)$*,* $(P_3, \leq_3)$ *be posets and* $\& \colon P_1 \times P_2 \to P_3$*,* $\leftarrow \colon P_3 \times P_2 \to P_1$*, be mappings, then* $(\&, \leftarrow)$*, is called an* adjoint pair *with respect to* P_1, P_2, P_3 *if:*

- $\&$ *is increasing in both arguments;*
- $\leftarrow$ *are increasing in the first argument and decreasing in the second;*
- $x \leq_1 z \leftarrow y$ *iff* $x \,\&\, y \leq_3 z$ *for all* $x \in P_1$*,* $y \in P_2$ *and* $z \in P_3$*;*

Now, we will pass to introduce a bit of fuzzy concept analysis. We will remind first the notions of Galois connection and concept [9,10].

Definition 8. *Let* ${}^{\downarrow} \colon P \to Q$ *and* ${}^{\uparrow} \colon Q \to P$ *be two maps between the posets* $(P, \leq)$ *and* $(Q, \leq)$*. The pair* $({}^{\uparrow}, {}^{\downarrow})$ *is called a* Galois connection *if:*

- $p_1 \leq p_2$ *implies* ${p_2}^{\downarrow} \leq {p_1}^{\downarrow}$ *for every* $p_1, p_2 \in P$*;*
- $q_1 \leq q_2$ *implies* ${q_2}^{\uparrow} \leq {q_1}^{\uparrow}$ *for every* $q_1, q_2 \in Q$*;*

– *$p \leq p^{\uparrow\downarrow}$ and $q \leq q^{\downarrow\uparrow}$ for all $p \in P$ and $q \in Q$;*

An interesting property of a Galois connection $(^\uparrow, ^\downarrow)$ is that $^\downarrow = {}^{\downarrow\uparrow\downarrow}$ and $^\uparrow = {}^{\uparrow\downarrow\uparrow}$.

Definition 9. *A pair (p, q) is called a* concept *if $p^\downarrow = q$ and $q^\uparrow = p$.*

If P and Q are lattices we have as well the following result:

Theorem 2. *[9] Let $(L_1, \preceq_1)$ and $(L_2, \preceq_2)$ be two complete lattices and $(^\uparrow, ^\downarrow)$ a Galois connection between them, then we have that the set $\mathcal{C} = \{(x, y) \mid x \in L_1, y \in L_2, x^\downarrow = y, y^\uparrow = x\}$ is a complete lattice with the following ordering $(x_1, y_1) \preceq (x_2, y_2)$ if and only if $x_1 \preceq_1 x_2$ (or equivalently $y_2 \preceq_2 y_1$), where the supremum and the infimum are given bellow:*

$$\bigwedge_{i \in I}(x_i, y_i) = \left(\bigwedge_{i \in I} x_i, (\bigvee_{i \in I} y_i)^{\uparrow\downarrow}\right)$$
$$\bigvee_{i \in I}(x_i, y_i) = \left((\bigvee_{i \in I} x_i)^{\downarrow\uparrow}, \bigwedge_{i \in I} y_i\right)$$

With all this notions now we can pass to the following section where we will present concept multilattices.

3 Fuzzy formal concept multilattices

When working in concept analysis theory we always have two sets A and B representing the attributes and the objects together with a relation between them. In order to reach the concept multilattices these ones will be evaluated in complete multilattices.

The first result we obtain concerning concept analysis in multilattices will be crucial for our purpose. We will denote by M_1^A and M_2^B the sets of all mappings from A to M_1 and from B to M_2 respectively.

Theorem 3. *Let $(M_1, \leq_1)$ and $(M_2, \leq_2)$ be two complete multilattices, A and B two sets and $(^\uparrow, ^\downarrow)$ a Galois connection between M_1^A and M_2^B. If $\{(g_i, f_i)\}_{i \in I}$ is a set of concepts we have that*

$$\mathit{multinf}\{{f_i}^\downarrow \mid i \in I\} \subseteq (\mathit{multisup}\{f_i \mid i \in I\})^\downarrow \tag{1}$$

$$\mathit{multinf}\{{g_i}^\uparrow \mid i \in I\} \subseteq (\mathit{multisup}\{g_i \mid i \in I\})^\uparrow \tag{2}$$

where $(\mathit{multisup}\{f_i \mid i \in I\})^\downarrow = \{f_{mult}^\downarrow \mid f_{mult} \in \mathit{multisup}\{f_i \mid i \in I\}\}$ and $(\mathit{multisup}\{g_i \mid i \in I\})^\uparrow$ is given similarly.

Proof. We will prove (1). Item (2) is proved in a similar way.

Let $g \in \text{multinf}\{f_i{}^{\downarrow} \mid i \in I\}$ we have that $g \leq_1 f_i{}^{\downarrow}$ for every $i \in I$. As $^{\uparrow}$ is decreasing we have that $f_i^{\downarrow\uparrow} \leq_2 g^{\uparrow}$, but the pair $(^{\uparrow}, ^{\downarrow})$ is a Galois connection so we have that:

$$f_i \leq_2 (f_i)^{\downarrow\uparrow} \leq_2 g^{\uparrow}$$

Then there is $f_{\text{mult}} \in \text{multisup}\{f_i \mid i \in I\}$ such that $f_{\text{mult}} \leq_2 g^{\uparrow}$. As $^{\downarrow}$ is decreasing we have that $g^{\uparrow\downarrow} \leq_1 f_{\text{mult}}^{\downarrow}$ using again that $(^{\uparrow}, ^{\downarrow})$ is a Galois connection we obtain that:

$$g \leq_1 g^{\uparrow\downarrow} \leq_1 f_{\text{mult}}^{\downarrow} \tag{3}$$

On the other hand, we have that $f_{\text{mult}} \in \text{multisup}\{f_i \mid i \in I\}$, so $f_i \leq_2 f_{\text{mult}}$ for every $i \in I$ then, as $^{\downarrow}$ is decreasing $f_{\text{mult}}^{\downarrow} \leq_1 f_i^{\downarrow}$, for every $i \in I$, then $f_{\text{mult}}^{\downarrow}$ is a lower bound of the set $\{f_i{}^{\downarrow} \mid i \in I\}$, but $g \in \text{multinf}\{f_i{}^{\downarrow} \mid i \in I\}$ and $g \leq_1 f_{\text{mult}}^{\downarrow}$, by (3). Therefore, by maximality of g we have that $g = f_{\text{mult}}^{\downarrow}$.

Thus, we have proved that for every $g \in \text{multinf}\{f_i{}^{\downarrow} \mid i \in I\}$ there is $f_{\text{mult}} \in \text{multisup}\{f_i \mid i \in I\}$ such that $g = f_{\text{mult}}^{\downarrow}$, which leads us to the result. □

We cannot get always the equality in this theorem as we can see in the next example:

Example 2. If we consider the multilattice of Fig. 1 and the folowing Galois connection, $^{\uparrow} = {}^{\downarrow} \colon M6 \to M6$ defined as:

$$\bot^{\uparrow} = \top \ ; \ a^{\uparrow} = b^{\uparrow} = c^{\uparrow} = c \ ; \ d^{\uparrow} = \bot \ ; \ \top^{\uparrow} = \bot$$

It is routine to prove that the pair $(^{\uparrow}, ^{\downarrow})$ is a Galois connection.

On one hand, we obtain that

$$\text{multinf}\{a^{\uparrow}, b^{\uparrow}\} = \text{multinf}\{c\} = c$$

However, on the other hand:

$$(\text{multisup}\{a, b\})^{\uparrow} = (\{c, d\})^{\uparrow} = \{c^{\uparrow}, d^{\uparrow}\} = \{c, \bot\}$$

which proves that we cannot get the equality always.

As a consequence of the previous theorem, we have that, given the set of all concepts $\mathcal{C} = \{(g, f) \mid f \in M_1^A, g \in M_2^B, g^{\uparrow} = f, f^{\downarrow} = g\}$, and the ordering defined as $(g_1, f_1) \leq (g_2, f_2)$ if and only if $g_1 \leq_1 g_2$ (if and only if $f_2 \leq_2 f_1$), then $(\mathcal{C}, \leq)$ is a complete multilattice which is a result similar to Theorem 2, but now with respect to multilattices.

Theorem 4. *If $(M_1, \leq_1)$ and $(M_2, \leq_2)$ be two complete multilattices, A and B two sets and $(^{\uparrow}, ^{\downarrow})$ a Galois connection between M_1^A and M_2^B, then we have that $(\mathcal{C}, \leq)$ is a complete multilattice where for every set of concepts $\{(g_i, f_i)\}_{i \in I}$:*

$$\mathit{multinf}\{(g_i, f_i)\} = (\mathit{multinf}\{g_i\}, (\mathit{multinf}\{g_i\})^{\uparrow}) \tag{4}$$

$$\mathit{multisup}\{(g_i, f_i)\} = ((\mathit{multinf}\{f_i\})^{\downarrow}, \mathit{multinf}\{f_i\}) \tag{5}$$

Proof. If we prove that they are concepts, then it is obvious that they are the multisuprema and the multinfima due to the definition of the ordering in $\mathcal{C}$.

By Theorem 3, we have that

$$\mathrm{multinf}\{g_i\} \subseteq (\mathrm{multisup}\{f_i\})^{\downarrow} \tag{6}$$

Hence, given $g \in \mathrm{multinf}\{g_i\}$, there exist $f \in \mathrm{multisup}\{f_i\}$, such that $g = f^{\downarrow}$. Therefore, since $({}^{\uparrow}, {}^{\downarrow})$ is a Galois connection, $g^{\uparrow\downarrow} = f^{\downarrow\uparrow\downarrow} = f^{\downarrow} = g$.

Consequently, it is trivial that they are concepts. For the multisuprema the proof is similar. The proof of coherence and the non-existence of anti-chains comes directly from the definition of the ordering consider. □

At this point we could think what would happen whether the set of objects or the set of attributes are evaluated in a lattice while the other in a multilattice. The answer to this is given by the following corollary.

Proposition 1. *Considering the framework of the previous theorem, if M_1 or M_2 is a lattice, then we have that $\mathcal{C}$ is a lattice.*

Proof. If M_1 is a lattice in the first equality of (4) the multinfimum becomes a singleton so it is indeed an infimum. Hence, every set has an infimum and so $(\mathcal{C}, \leq)$ is a complete lower semilattice. Therefore, we only have to prove that there is a maximum element $\top_{\mathcal{C}}$ in $\mathcal{C}$.

Let $g_\top \in M_1^A$ the map which sends every element of A to the maximum element $\top$ of M_1 and consider the pair $(g_\top, g_\top^{\uparrow})$. If we prove that it is a concept then we have finished, since it is obvious that this element would be the maximum element in $\mathcal{C}$.

We only have to prove that $g_\top = g_\top^{\uparrow\downarrow}$. As $({}^{\uparrow}, {}^{\downarrow})$ a Galois connection we have that $g_\top \leq_1 g_\top^{\uparrow\downarrow}$ and, as $g_\top(a) = \top$ for every element $a \in A$, we have that the equality holds, i.e., $g_\top = g_\top^{\uparrow\downarrow}$.

The proof for M_2 being a lattice is similar. □

The following section introduces a simple and particular context where we can get a Galois connection from an adjoint pair what allows us to obtain a concept multilattice.

4 A worked out example

The multilattice considered for the calculation in this example is the one given in Fig. 2 together with the following adjoint pair $(\&, \leftarrow)$.

$$x \,\&\, y = \begin{cases} x & \text{if } y = \top \\ y & \text{if } x = \top \\ \bot & \text{if } x \in \{\bot, b\} \text{ or } y \in \{\bot, b\} \\ a & \text{otherwise} \end{cases}$$

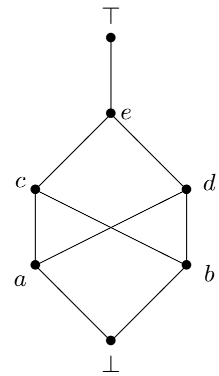

Fig. 2. Multirretículo M6*

$$z \leftarrow y = \begin{cases} \top & \text{if } y \leq z \\ z & \text{if } y = \top \\ b & \text{if } y \notin \{\bot, b, \top\} \text{ and } z \in \{\bot, b\} \\ e & \text{otherwise} \end{cases}$$

It is routine calculation that $(\&, \leftarrow)$ is, indeed, an adjoint pair, in which $\&$ is commutative.

Imagine that we are going to travel to a city and we have to decide which hotel is the best for us. In this example, in order to no complicate the calculation we will taking into account seven different hotels, as objects, and two attributes, which will be *price* and *situation*. Hence, we have as set of objects $B = \{H1, H2, H3, H4, H5, H6, H7\}$ and as set of attributes $A = \{\text{price}, \text{situation}\}$, both evaluated in $M6^*$ and the $M6$-fuzzy relation, $R\colon A \times B \to M6^*$, between them, defined in Table 1

Table 1. Relation R

R	price	situation
H1	d	$\bot$
H2	c	a
H3	$\top$	b
H4	a	d
H5	b	e
H6	a	b
H7	d	c

Evaluating the hotels in a multilattice comes from the idea that the hotels are ordered thinking of the number of stars they have. We can state, for example that any four-star hotel is better than any three-star hotel, but if both hotels are four-star ones we cannot distinguish between them at the beginning.

In the case of the situation, we have that we can say one situations are better than other but we cannot compare a situations that are for example one kilometer from the downtown but in different directions.

In the case of prizes, happens more or less the same because we cannot distinguish between prizes which are very similar.

If we see the relationship we have that $R(H5, \text{price}) = b$, $R(H6, \text{price}) = a$ means that the fifth and the sixth hotels have more or less the same prices but we cannot decide which is best taking into account only their prizes.

For these reasons we have chosen multilattices for their evaluation.

We are trying to choose a hotel to stay in according to our preferences in prizes and situation.

It is easy to check that for the adjoint pair $(\&, \leftarrow)$ and for any mapping $f\colon A \to M6^*$ or $g\colon B \to M6^*$ the following sets has infimum.

$$\{R(a,b) \leftarrow g(b) \mid b \in B\}$$
$$\{R(a,b) \leftarrow f(a) \mid a \in A\}$$

Hence, we can define the next Galois connection

$$g^{\uparrow}(a) = \inf\{R(a,b) \leftarrow g(b) \mid b \in B\}$$
$$f^{\downarrow}(b) = \inf\{R(a,b) \leftarrow f(a) \mid a \in A\}$$

The proof of $(\downarrow, \uparrow)$ being a Galois connection follows directly from the existence of the infimum of these sets, that $\&$ is commutative and that the implication are decreasing in the second argument.

Therefore, from Theorem 4 we have an fuzzy concept multilattice and if our preferences are the following $g(\text{price}) = a$ and $g(\text{situation}) = d$ we have that for $H1$.

$$g^{\uparrow}(H1) = \inf\{d \leftarrow a, \bot \leftarrow d\} = \inf\{\top, b\} = b$$

And for the others:

$$g^{\uparrow}(H2) = e\ ,\ g^{\uparrow}(H3) = b\ ,\ g^{\uparrow}(H4) = \top\ ,\ g^{\uparrow}(H5) = b\ ,\ g^{\uparrow}(H6) = b\ ,\ g^{\uparrow}(H7) = e$$

On the other hand we have that

$$\begin{aligned} f^{\downarrow\uparrow}(\text{price}) &= \inf\{d \leftarrow b, c \leftarrow e, \top \leftarrow b\ , a \leftarrow \top\, b \leftarrow b, a \leftarrow b, a \leftarrow e\} \\ &= \inf\{\top, \top, a, \top, e, e\} = a \end{aligned}$$

In a similar way we obtain that

$$f^{\downarrow\uparrow}(\text{situation}) = d$$

Thus, according to out preference stablished by f, we have that our best choice is $H4$, although $H2$ and $H7$ are really good ones too.

5 Conclusions and future work

A first approach to fuzzy formal concept multilattices has been presented. This paradigm arises as a more flexible setting than formal concept analysis framework, as the introduced motivating example shows.

Moreover, several properties have been proved. For example, we have checked that the concepts in the new framework form a complete multilattice and that if we impose that one of the set of attributes or objects are evaluated in a lattice and the other in a multilattice, then we obtain a complete lattice.

In the future, we will study general Galois connections which allows us to get more concept multilattices. We will focus as well, when we have these Galois connections, on getting a representation theorem for them to be able to get which multilattices are isomorphic to concept multilattices.

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