

# Annotating Lattice Orbifolds with Minimal Acting Automorphisms

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**Abstract** Context and lattice orbifolds have been discussed by M. Zickwolff [1,2], B. Ganter and D. Borchmann[3,4]. Preordering the folding automorphisms by set inclusion of their orbits gives rise to further development. The minimal elements of this preorder have a prime group order and any group element can be dissolved into the product of group elements whose group order is a prime power. This contribution describes a way to compress an orbifold annotation to sets of such minimal automorphisms. This way a hierarchical annotation is described together with an interpretation of the annotation. Based on this annotation an example is given that illustrates the construction of an automaton for certain pattern matching problems in music processing.

**Key words:** formal concept lattice, lattice orbifold, annotation, automorphism group

## 1 Introduction

Lattice orbifolds have been described by Monika Zickwolff [1,2] as a useful tool for the compression of formal concept lattices. Daniel Borchmann has extended this theory in his diploma thesis to context orbifolds [3,4]. A binary relation orbifold can be considered as a mathematical structure on the sets of orbits of a given group of automorphisms of a binary relation structure that allows to reconstruct the original relation. Thus, it can provide deeper insight into the structure of such a relation. On the other hand it provides the means for compressing a relational structure in a way that preserves the possibility for certain algorithms to act on it.

In the work of Zickwolff, Ganter and Borchmann together with the theory also a method of data compression by means of the stabilisers in the automorphism group has been provided. This abridged annotation contains the automorphisms that violate a certain kind of symmetry which can be described by the stabilisers of the equivalence classes. Thus, it provides an insight how the lattice violates the symmetry reflected by the folding group.

The latter approach starts from a global view at the orbifold and cuts out redundant information treating all nodes in the Hasse diagram equally. Properties like direction are not used for this kind of annotation.

The work presented here, starts from a local view (a pair of neighbours) and uses both direction and transitivity of the relation to minimise the annotation. In this way it spares the necessity to save the stabilisers increasing the information provided by the edge annotation in comparison with the classical abridged annotation.

After some theoretical section an example is given that provides further insight into applications of the theory provided in this article.

## 2 Preliminaries

If not stated otherwise algebraic structures are denoted by double stroke letters, the base set of an algebraic structure  $\mathbb{A}$  by the same letter  $A$  in normal font.  $\mathfrak{Aut} \mathbb{A}$  is its automorphism group and  $\mathbb{1} := (\{(1)\}, \cdot, (1))$  the trivial group. For any permutation group  $\mathbb{G}$  on a set  $A$  we denote the set of its orbits by  $A \parallel \mathbb{G} := \{x^{\mathbb{G}} \mid x \in A\}$ . Obviously, for any group element  $g \in G$  and any orbit  $U \in A \parallel \mathbb{G}$  also its adjoint  $gUg^{-1}$  is an orbit. Throughout this paper we refer to the set of volatile points of an automorphism  $g \in \mathfrak{Aut} \mathbb{A}$  as  $\mathfrak{Var} g := \{x \in M \mid x^g \neq x\}$  and to its set of fixed points using the notation  $\mathfrak{Fix} g := \{x \in M \mid x^g = x\}$ . Obviously, for any element  $g \in \mathfrak{Aut} \mathbb{A}$  the equation  $A = \mathfrak{Fix} g \cup \mathfrak{Var} g$  holds. We say that a permutation  $g \in G$  acts semiregular on a set  $M \subseteq A$ , iff  $M \setminus \mathfrak{Var} g \in \{\emptyset, M\}$  is true. Note that  $\mathfrak{Var} g$  is not restricted to be a subset of  $M$ .

If  $(M, \leq)$  is an ordered set and  $N \subseteq M$  then the corresponding order ideal is defined by the set  $\downarrow_{\leq} N := \{1 \in M \mid \exists y \in N : x \leq y\}$  and we write  $\downarrow_{\leq} x$  for  $\downarrow_{\leq} \{x\}$  if the context is clear. The neighbourhood relation is denoted by the symbol  $\prec$ .

As defined in [3] an orbifold of an ordered set  $(M, \leq_M)$  will be denoted by a triple  $(M \parallel \mathbb{G}, \leq, \lambda)$  where  $x^{\mathbb{G}} \leq y^{\mathbb{G}} :\Leftrightarrow \exists g \in G : x \leq_M y^g$  and  $\lambda$  is an annotational function which carries additional information that allows to reconstruct the relation  $\leq_M$ . The most generic choice of  $\lambda$  is defined in the same article as a mapping  $\lambda : (M \parallel \mathbb{G}) \times (M \parallel \mathbb{G}) \rightarrow \mathfrak{P} G$  which fulfils the condition  $\lambda(x^{\mathbb{G}}, y^{\mathbb{G}}) = \{g \in G \mid x \leq_M y^g\}$  for any  $x, y \in M$ . In this setting the stabiliser  $\mathbb{G}_x$  of an Element  $x$  can be retrieved from its annotation  $\lambda(x^{\mathbb{G}}, x^{\mathbb{G}})$ .<sup>1</sup>

A simplified description of order orbifolds is defined utilizing the fact

$$\lambda(x^{\mathbb{G}}, y^{\mathbb{G}}) \setminus \bigcup_{x^{\mathbb{G}} < z^{\mathbb{G}} < y^{\mathbb{G}}} \lambda(x^{\mathbb{G}}, z^{\mathbb{G}}) \cdot \lambda(z^{\mathbb{G}}, y^{\mathbb{G}}) = \bigcup_{\substack{g \in \lambda(x^{\mathbb{G}}, y^{\mathbb{G}}) \\ g \notin \lambda(x^{\mathbb{G}}, z^{\mathbb{G}}) \cdot \lambda(z^{\mathbb{G}}, y^{\mathbb{G}}) \\ x^{\mathbb{G}} < z^{\mathbb{G}} < y^{\mathbb{G}}}} \mathbb{G}_x \cdot g \cdot \mathbb{G}_y.$$

This consists of a transversal  $T$  of  $M \parallel \mathbb{G}$  and an abridged annotation function  $\lambda_{abr} : T \times T \rightarrow \mathfrak{P} G$ , where  $\lambda_{abr}(x, y)$  is a set of double coset representatives, i. e. it is a minimal set such that  $\mathbb{G}_x \cdot \lambda_{abr}(x, y) \cdot \mathbb{G}_y = \lambda(x, y)$ .

It is well known that orbits of automorphisms of a finite ordered set are always antichains in this set.

<sup>1</sup> For other settings see [1,2,4].

### 3 Minimal Acting Automorphisms

Let  $\mathbb{A}$  be an algebraic structure. The orbits of the cyclic subgroups of  $\mathfrak{Aut} \mathbb{A}$  can be used to preorder the automorphism group. If we refer to some group  $\mathbb{G}$  or its base set  $G$  without further notice it is always meant to be  $\mathbb{G} \leq \mathfrak{Aut} \mathbb{A}$ .

**Lemma 1.** *Let  $\mathfrak{Aut} \mathbb{A}$  the automorphism group of a finite algebraic structure  $\mathbb{A}$ . Then the binary relation  $\sqsubseteq \subseteq \mathfrak{Aut} \mathbb{A} \times \mathfrak{Aut} \mathbb{A}$  defined by*

$$g \sqsubseteq h :\Leftrightarrow \forall U \in (A \parallel \langle g \rangle) \exists U' \in (A \parallel \langle h \rangle) : U \subseteq U'$$

is a preorder.

*Proof.* This follows directly from the definition: Reflexivity is obvious as the equation  $A \parallel \langle g \rangle = A \parallel \langle g \rangle$  holds. Given three automorphisms  $f, g, h \in \mathfrak{Aut} \mathbb{A}$  such that for each orbit  $U \in (A \parallel \langle f \rangle)$  there exists an orbit  $U' \in (A \parallel \langle g \rangle)$  with  $U \subseteq U'$ . If the same condition is true for the pair  $(g, h)$  we can find an orbit  $U'' \in (A \parallel \langle h \rangle)$  such that  $U \subseteq U' \subseteq U''$ . Thus transitivity holds, too.  $\square$

As in any cyclic subgroup the implication  $\langle g^n \rangle \subseteq \langle g \rangle \Rightarrow x^{\langle g^n \rangle} \subseteq x^{\langle g \rangle}$  holds, we can fix the following corollary:

**Corollary 1.** *For any group element  $g \in \mathfrak{Aut} \mathbb{A}$  and any natural number we get:  $g^n \sqsubseteq g$*

In particular, this means that  $g \in \mathfrak{Aut} \mathbb{A}$  and  $n \in \mathbb{N}$  imply  $\mathfrak{Var} g^n \subseteq \mathfrak{Var} g$ .

On the other hand, the so defined relation  $\sqsubseteq$  is usually no order relation as for any  $g \in \mathfrak{Aut} \mathbb{A}$  we have  $A \parallel \langle g \rangle = A \parallel \langle g^{-1} \rangle$ , but in general the equation  $g = g^{-1}$  does not hold.

If  $A$  is finite, then the preordered set  $(\mathfrak{Aut} \mathbb{A}, \sqsubseteq)$  has minimal elements.

**Definition 1.** *Let  $\mathbb{A}$  be a finite algebraic structure. The minimal elements of the preordered set  $(\mathfrak{Aut} \mathbb{A}, \sqsubseteq)$  are called automorphisms with minimal action.*

**Corollary 2.** *Let  $g$  be minimal in  $(G, \sqsubseteq)$ , then the cyclic group  $\langle g \rangle$  acts semiregular on  $\mathfrak{Var} g$ .*

*Proof.* Suppose  $\langle g \rangle$  does not act semiregular on  $\mathfrak{Var} g$ . Then there exist elements  $\exists x, y \in \mathfrak{Var} g$  and a positive integer  $n \in \mathbb{N} \setminus \{0\}$  such that  $x^{g^n} = x, y^{g^n} \neq y$ . This implies  $x^{\langle g^n \rangle} = \{x\} \neq x^{\langle g \rangle}$ . Thus,  $x^{\langle g^n \rangle} \not\subseteq A \parallel \langle g \rangle$ . Consequently,  $g^n \not\sqsubseteq g$  and  $g \not\sqsubseteq g^n$ . Thus,  $g$  is not minimal.  $\square$

**Corollary 3.** *Let  $g \in G$  be minimal in  $(G, \sqsubseteq)$ . Then for any element  $h \in G$  :  $huh^{-1}$  is minimal, too.*

*Proof.* Suppose the existence of an element  $v \in G$  such that the set of its orbits  $A \parallel \langle v \rangle$  is a refinement of  $A \parallel \langle gug^{-1} \rangle$ . Then,  $A \parallel \langle g^{-1}vg \rangle = (A \parallel \langle v \rangle)g$  is a refinement of  $A \parallel \langle u \rangle$ , as  $A = A^{g^{-1}}$ . This means that  $u$  is not minimal.  $\square$

The set containing the minimal nontrivial automorphisms of  $(G, \sqsubseteq)$  will be denoted by  $\mathfrak{Min}(G, \sqsubseteq)$ , in particular we define

$$(1) \in \mathfrak{Min}(G, \sqsubseteq) \text{ iff } \mathfrak{Min}(G, \sqsubseteq) \setminus \{(1)\} = \emptyset \text{ and } G \neq \emptyset.$$

**Corollary 4.** *The subgroup  $\langle \mathfrak{Min}(G, \sqsubseteq) \rangle$  is a normal subgroup in  $\mathbb{G}$ .*

Hall's theorem [5] tells us that we can dissolve each cyclic subgroup of  $\mathbb{G}$  into a product of cyclic groups whose orders are prime powers. Thus, we can generate each cyclic group by a set of elements with pairwise coprime orders. This leads us to the following corollary:

**Corollary 5.** *Let  $\mathbb{G} \leq \mathfrak{Aut} \mathbb{A}$ . Then the set*

$$P := \{g \in G \mid |\langle g \rangle| \text{ is a prime power}\}$$

*is a generating set of  $\mathbb{G}$ .*

For the construction of the Sylow groups, we can use the following lemma:

**Lemma 2.** *Let  $g \in \mathfrak{Aut} \mathbb{A}$  an automorphism of finite order  $n \in \mathbb{N} \setminus \{0\}$  and  $g_1, g_2 \in \langle g \rangle$  with  $|\langle g_1 \rangle| = m_1$  and  $|\langle g_2 \rangle| = m_2$ . Then  $\langle g^{\gcd(n/m_1, n/m_2)} \rangle \leq \langle g_1, g_2 \rangle$ .*

*Proof.* As cyclic groups are Abelian, there exist integers  $a, b \in \mathbb{Z}$  such that  $\gcd(\frac{n}{m_1}, \frac{n}{m_2}) = a \frac{n}{m_1} + b \frac{n}{m_2}$ . Since  $g_1 \in \langle g^{n/m_1} \rangle$  and  $g_2 \in \langle g^{n/m_2} \rangle$ , w.l.o.g. we can assume  $g_1 = g^{n/m_1}$  and  $g_2 = g^{n/m_2}$ . Thus, we get  $g^{\gcd(n/m_1, n/m_2)} = g_1^a g_2^b$ .  $\square$

Consequently, the generating set of Corollary 5 contains all minimal elements  $\mathfrak{Min}(G, \sqsubseteq)$ .

**Lemma 3.** *Let  $\mathbb{G} \leq \mathfrak{Aut} \mathbb{A}$  be a finite automorphism group. Then the elements of  $\mathfrak{Min}(\mathbb{G}, \sqsubseteq)$  have prime order.*

*Proof.* Let  $|\langle g \rangle| = p^n$  where  $p$  is prime. Then  $|\langle g^{p^{n-1}} \rangle| = p$ . As  $\langle g \rangle$  is cyclic, its order is the least common multiple of the sizes of its orbits. Thus, it has at least one orbit of size  $p^n$ , while the orbits in  $\langle g^{p^{n-1}} \rangle$  have either one or  $p$  elements. Corollary 1 tells us, that the minimal Elements of  $(G, \sqsubseteq)$  are those of prime orders.  $\square$

## 4 Automorphisms of ordered sets

Let  $\mathbb{G} \leq \mathfrak{Aut} \mathbb{A}$  and  $U_1, U_2 \leq \mathbb{G}$  such that  $U_1 \cdot U_2 = \mathbb{G}$ . Considering the implied action of  $\mathbb{G}$  on  $\mathfrak{P} A$ , a straight forward calculation shows for all  $X \subseteq \mathfrak{P} A$  that  $X \in (A \parallel U_1) \parallel U_2$  iff  $\bigcup X \in A \parallel \mathbb{G}$ .

Let us consider some additional properties of automorphisms of finite lattices.

**Definition 2.** Let  $(M, \leq)$  be an ordered set,  $\mathbb{G} \leq \mathfrak{Aut}(M, \leq)$ , and  $x \in M$ . The set

$$\mathbb{G}_{\downarrow \leq x} := \{g \in \mathbb{G} \mid \mathfrak{Var} g \cap \downarrow \leq x = \emptyset\} \quad (1)$$

is called downwards stabiliser of  $x$  in  $\mathbb{G}$ .

Obviously the downwards stabiliser of a maximal element in a complete lattice is  $\mathbb{1}$ . In fact  $\mathbb{G}_{\downarrow \leq x}$  is a subgroup of the stabiliser  $\mathbb{G}_x$  of  $x$ .

**Corollary 6.** Let  $\mathbb{V} = (V, \leq)$  be a finite lattice ordered set. And let  $x \in V$  while  $y, z \in V$  are upper neighbours of  $x$ . Furthermore, if there exist two automorphisms  $g, h \in \mathbb{G}_{\downarrow \leq x}$  with the property  $z^g = y = z^h$ , then the equation  $(\downarrow \leq y)^{h^{-1}g} = \downarrow \leq y$  holds.

*Proof.* We know that  $y^{h^{-1}g} = y$ . So for any  $a \in \downarrow \leq y$  we know  $a^{h^{-1}g} \leq y$  as  $g$  and  $h$  are automorphisms.  $\square$

Thus, if two elements are in the same orbit their downwards stabilisers are related by conjugation. This proves the following lemma:

**Lemma 4.** Let  $\mathbb{V} = (V, \leq)$  a finite lattice ordered set,  $\mathbb{G} \leq \mathfrak{Aut} \mathbb{V}$ , and let  $x, y, z \in V$  while  $y$  and  $z$  are upper neighbours of  $x$ . Any automorphism  $g$  with  $z^g = y$  is an automorphism mapping  $V \setminus \downarrow \leq z$  to  $V \setminus \downarrow \leq y$ , while the equation  $\mathbb{G}_{\downarrow \leq y} = g^{-1}\mathbb{G}_{\downarrow \leq z}g$  holds.

*Proof.* Let  $g \in \mathbb{G}_{\downarrow \leq x}$  with  $z^g = y$  and let  $h \in \mathbb{G}_{\downarrow \leq z}$ . Then for any  $a \in V \setminus \downarrow \leq z$  and any  $b \in V \setminus \downarrow \leq y$  we get  $a^g \in V \setminus \downarrow \leq y$  and  $b^{g^{-1}} \in V \setminus \downarrow \leq z$  as  $g$  is an automorphism. As  $V \setminus \downarrow \leq y$  and  $V \setminus \downarrow \leq z$  are isomorphic by  $g$ , for any automorphism  $h \in \mathbb{G}_{\downarrow \leq z}$  the mapping  $f := g^{-1}hg$  is an automorphism on  $V \setminus \downarrow \leq y$  and even  $f \in \mathbb{G}_{\downarrow \leq y}$ , as it is constant for any  $c \in \downarrow \leq y$ . Finally, we get  $gfg^{-1} = h$ . Thus,  $\mathbb{G}_{\downarrow \leq z}$  is a conjugate of  $\mathbb{G}_{\downarrow \leq y}$ . The other direction of the implication is obvious.  $\square$

As an immediate conclusion, we get that the downwards stabiliser of an element is contained in the union of the downwards stabilisers of its upper neighbours:

**Corollary 7.** Let  $\mathbb{V} = (V, \leq)$  a finite lattice ordered set,  $\mathbb{G} \leq \mathfrak{Aut} \mathbb{V}$ , and let  $x \in V$  and  $N = \{y \in V \mid x \prec y\}$  the set of upper neighbours of  $x$ . Let further  $T$  a transversal of  $N \parallel \mathbb{G}$  and  $S \subseteq \mathbb{G}$  such that  $T^S = N$ . Then  $S \cdot \bigcup_{t \in T} \mathbb{G}_{\downarrow \leq t} \subseteq \mathbb{G}_{\downarrow \leq x}$ .

In other words: At any point in the lattice we can restrict ourselves to a local view. These considerations can be easily extended to finite ordered sets.

## 5 Orbifolds

In this section we define an orbifold representation using minimisations according to the preorder discussed in Section 3.

**Definition 3.** Let  $\lambda : T \times T \rightarrow \mathfrak{P}G$  be a mapping that assigns to each pair of elements from a finite set  $T$  to a subset of another set  $G$ . A contiguous chain of  $\lambda$  from  $x \in T$  to  $y \in T$  is defined as a subset  $C \subseteq T$  such that the relation  $\rho := \{(z, z') \in T \times T \mid \lambda(z, z') \neq \emptyset\}$  forms the neighbourhood relation of a linear order with minimal element  $x$  and maximal element  $y$ . The set of all such contiguous chains of  $\lambda$  between two elements  $x$  and  $y$  will be denoted by  $\mathfrak{C}_{T,\lambda}(x, y)$ .

Further let the relation  $\prec' \subseteq T \times T$  be defined by  $x \prec' y \Leftrightarrow \exists g \in G : x \prec y^g$ . Using the non-commutative complex product  $\prod$  in largest-left order, the operator  $\Lambda_\lambda : T \times T \rightarrow \mathfrak{P}G$  is defined by  $\Lambda(x, x) = \mathbb{1}$ , and for  $x \neq y$  by

$$\Lambda_\lambda(x, y) := \left\langle \bigcup \left\{ \prod_{i=2}^{|C|} \lambda(z_{i-1}, z_i) \mid \begin{array}{l} C \in \mathfrak{C}_{T,\lambda}(x, y), z_{i-1} \prec z_i, \\ \{z_1, z_2, \dots, z_{|C|}\} = C \end{array} \right\} \right\rangle \cap \mathbb{G}_{x,y}. \quad (2)$$

Theorem 7 will provide us with another kind of annotation of an orbifold:

**Definition 4.** Let  $\mathbb{V} = (V, \leq)$  be a finite lattice ordered set,  $\mathbb{G} \leq \mathfrak{Aut} \mathbb{V}$ ,  $T$  a transversal of  $V \parallel \mathbb{G}$ , and the relation  $\prec'$  defined as above.

A mapping  $\lambda_{hier} : T \times T \rightarrow \mathfrak{P}G$  is called hierarchical annotation (of  $V$ ,  $T$  and  $\mathbb{G}$ ), if it fulfils the following conditions for all  $x, y \in T$ :

1.  $\lambda_{hier}(x, y) \subseteq \mathbb{G}_{\downarrow \leq x}$ ,
2.  $\lambda_{hier}(x, y) = \emptyset$  if  $\forall y' \in y^{\mathbb{G}} : x \not\prec' y'$ , and
3.  $x \prec' y$  implies  $y^{\lambda_{hier}(x,y)} = y^{\mathbb{G}_{\downarrow \leq x}}$
4.  $y^{\lambda_{hier}(x,y) \Lambda_{\lambda_{hier}}(0,x)} = y^{\mathbb{G}_x}$ .

**Corollary 8.** Let  $\mathbb{V} = (V, \leq)$  be a finite lattice ordered set,  $\mathbb{G} \leq \mathfrak{Aut} \mathbb{V}$ , and  $\lambda$  a hierarchical annotation. Then the equation  $\langle \bigcup_{x,y \in T} \lambda(x, y) \rangle \leq \mathbb{G}$  holds.

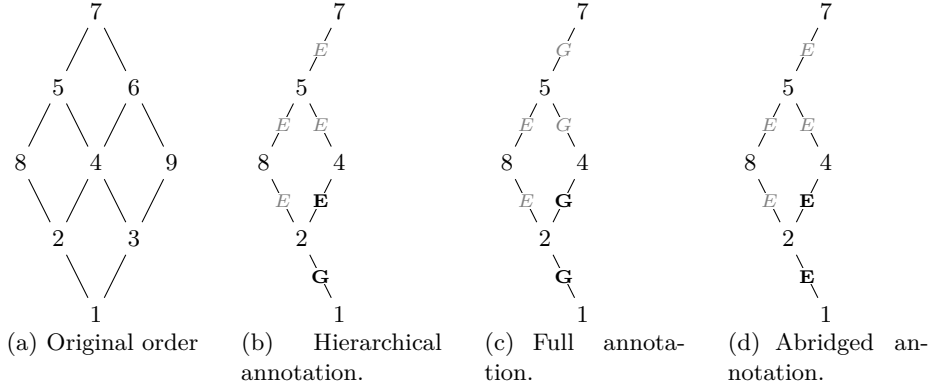
*Example 1.* Figure 1 shows a simple example of a lattice and its orbifold annotated with three different annotations. Besides the hierarchical annotation the annotations from Borchmann, Ganter and Zickwolff [1,2,3,4] have been included.

Before we explore some basic properties of hierarchical annotations we must assure their existence:

**Lemma 5.** Let  $\mathbb{V} = (V, \leq)$  be a finite lattice ordered set,  $\mathbb{G} \leq \mathfrak{Aut} \mathbb{V}$  a group of automorphisms, and  $T$  a transversal of  $V \parallel \mathbb{G}$ . Then there exists a hierarchical annotation  $\lambda : T \times T \rightarrow \mathfrak{P}G$ .

*Proof.* For any  $x, y \in T$  and any  $z \in V$  with  $x \prec y$ ,  $x \prec z$ , if  $z \in y^{\mathbb{G}_{\downarrow \leq x}}$  we fix an arbitrary  $g_{x,y,z} \in \mathbb{G}_{\downarrow \leq x}$  such that  $z = y^{g_{x,y,z}}$ . In that case we define  $\lambda_1(x, y) := \{g_{x,y,z} \mid z \in y^{\mathbb{G}_{\downarrow \leq x}}\}$ . If  $z \in y^{\mathbb{G}_x} \setminus y^{\mathbb{G}_{\downarrow \leq x}}$  and  $\mathbb{G}_{\downarrow \leq x} = \emptyset$  then the orbits of  $y$  are predefined by all automorphisms that act below  $x$  thus, we define  $\lambda_1(x, y) := \mathbb{1}$ .

In any case where  $z \in y^{\mathbb{G}_x} \setminus y^{\mathbb{G}_{\downarrow \leq x}}$ , there exists an automorphism  $h_{x,y,z} \in \mathbb{G}_x$  such that  $z \in y^{\lambda_1(x,y) h_{x,y,z}}$ . Then there exist two elements  $\hat{x}_z, \hat{y}_z \in \downarrow_{\leq'} x$  such



**Figure 1.** The lattice is folded by the group  $G = \{(1), (23)(56)(89)\}$ . As you can see in 1(b),  $\mathbb{G}_4 = G$ , but  $\lambda_{hier}(4, 5) = \{(1)\}$ . The singleton  $E$  is defined by  $E := \{(1)\}$ .

that  $\hat{x}_z \prec' \hat{y}_z$  and there is an automorphism  $h_{x,y,z} \in \mathbb{G}_{\downarrow \leq \hat{x}_z} \setminus \mathbb{G}_{\hat{y}_z}$  and another automorphism  $\hat{h}_{x,y,z} \in \Lambda(\hat{x}, x)$ . Using this we define a second preliminary annotation  $\lambda_2(\hat{x}_z, \hat{y}_z, x, y) := \{h_{x,y,z} \hat{h}_{x,y,z}^{-1} \mid z \in y^G \setminus y^{\mathbb{G}_{\downarrow \leq x}}\}$ .

For all other combinations of elements  $\hat{x}, \hat{y}, x, y$  we set  $\lambda_1(x, y) := \emptyset$  and  $\lambda_2(\hat{x}, \hat{y}, x, y) := \emptyset$ . Finally we define:

$$\lambda_{hier}(x, y) := \lambda_1(x, y) \cup \bigcup_{\tilde{x}, \tilde{y} \in T, \tilde{z} \in \tilde{y}^{\mathbb{G}_x}} \lambda_2(x, y, \tilde{x}, \tilde{y}, \tilde{z}).$$

Obviously such a function exists and fulfils the conditions of Definition 4.  $\square$

Now, as we know how to describe the automorphism group  $\mathbb{G} \leq \mathfrak{Aut} \mathbb{V}$  by means of automorphisms acting locally, we will use automorphisms that are minimal under certain restrictions. The corresponding operator is defined as follows:

**Definition 5.** Let  $\mathbb{V} = (V, \leq)$  be a finite lattice ordered set, and  $\mathbb{G} \leq \mathfrak{Aut} \mathbb{V}$  an automorphism group of  $\mathbb{V}$ , while  $U \subseteq G$  is one of its subsets. For any  $x, y, z \in V$  the elements of the set

$$\mathfrak{Min}_{x,y \mapsto z} U := \mathfrak{Min}_{\square} \{g \in U \mid x^g = x, y^g = z\} \quad (3)$$

are called minimal annotating automorphisms (fixing  $x$  and mapping  $y$  to  $z$ ). The elements of the set

$$\mathfrak{Min}_{\downarrow x, y \mapsto z} U := \begin{cases} \mathfrak{Min}_{x,y \mapsto z} U \cap \mathbb{G}_{\downarrow \leq x} & \mathfrak{Min}_{x,y \mapsto z} U \cap \mathbb{G}_{\downarrow \leq x} \neq \emptyset \\ \mathbf{1} & \mathfrak{Min}_{x,y \mapsto z} U \cap \mathbb{G}_{\downarrow \leq x} = \emptyset \text{ and} \\ \emptyset & \mathfrak{Min}_{x,y \mapsto z} U \neq \emptyset \\ & \text{else} \end{cases} \quad (4)$$

are called upper minimal automorphisms.

For applications it would be interesting to have an annotation that consists only of minimal acting automorphisms. Unfortunately, that is not generally possible. Nevertheless when  $\mathcal{Irrred}_{\leq}^{\vee} : V \rightarrow \mathfrak{P}V$  maps each element to the set of supremum irreducible elements less or equal to it, we can prove the following lemma:

**Lemma 6.** *For any finite lattice ordered set  $\mathbb{V} = (V, \leq)$ , any automorphism group  $\mathbb{G} \leq \mathcal{A}ut(V, \leq)$ , and any transversal  $T \subseteq V$  of  $V \parallel \mathbb{G}$  there exists a hierarchical annotation that consists of upper minimal automorphisms, if for all elements  $x, y \in T$  with  $x \prec y$  and  $z \in \mathcal{Irrred}_{\leq}^{\vee} y \setminus \mathcal{Irrred}_{\leq}^{\vee} x$  the following condition holds:*

$$(\mathcal{Irrred}_{\leq}^{\vee} y \setminus \mathcal{Irrred}_{\leq}^{\vee} x)^{\mathbb{G}} = z^{\mathbb{G}} \quad (5)$$

*Proof.* It is a well-known fact, that for each automorphism  $g \in G$  and every element  $x \in V$  the equation  $x^g = \bigvee ((\mathcal{Irrred}_{\leq}^{\vee} x)^g)$ .

Let  $x \prec' y$  a pair of neighbours in  $(T, \leq')$  and  $x \prec z$  a pair of neighbours in  $(V, \leq)$  such that  $z \in y^{\mathbb{G}}$ . Then we can modify the proof of Lemma 5 with the following refinements:

1. If  $y \in \mathcal{Irrred}_{\leq}^{\vee} y$  then choose for any  $z \in y^{\mathbb{G}_{\downarrow \leq x}}$  a minimal automorphism  $g_{x,y,z} \in \mathcal{M}in_{x,y \rightarrow z} \mathbb{G}$  and define  $\lambda_1$  as in Lemma 5.
2. For  $y \in \mathcal{Irrred}_{\leq}^{\vee} y$  and  $z \notin y^{\mathbb{G}_{\downarrow \leq x}}$  there exist an automorphism  $g \in \mathbb{G}_x \setminus \mathbb{G}_{\downarrow \leq x}$  and an automorphism  $h \in \lambda_1(x, y)$  such that  $z = y^{hg}$  where  $g \neq (1)$ . In that case the action of  $\langle g \rangle$  on  $z$  depends on the action on  $\downarrow_{\leq} x$ . As  $(\downarrow_{\leq} x)^{\langle g \rangle} \subseteq \downarrow_{\leq} x$  and the action of  $\langle g \rangle$  on  $\downarrow_{\leq} x$  is defined by the irreducibles also the action on  $z$  depends on the irreducibles below it (everything else we have already collected in  $\lambda_1(x, y)$ ). Let  $0 = \hat{x}_0 \prec' \hat{x}_1 \prec' \dots \prec' \hat{x}_l = x$  be a maximal chain from 0 to  $x$  of elements of  $V$ . Then there exists a chain  $x_0 \prec x_1 \prec \dots \prec x_l$  such that  $x_i \in \hat{x}_i^{\mathbb{G}}$ . For each pair  $(x_i, x_{i+1})$  we define  $I(x_i, x_{i+1}) := \mathcal{Irrred}_{\leq}^{\vee} x_{i+1} \setminus \mathcal{Irrred}_{\leq}^{\vee} x_i$ . Let  $g_0 = (1)$ . Given  $\hat{x}_i^{g_i} = x_i$  chose a minimal automorphism  $m_i$  from  $\lambda(\hat{x}_i, \hat{x}_{i+1})$  that maps  $\hat{x}_i$  to  $x_{i+1}^{g_i^{-1}}$ . This is always possible as  $|I(x_i, x_{i+1})^{\mathbb{G}} \cap T| = 1$ . Then define  $g_{i+1} := m_i g_i$ . Finally we get an automorphism  $g_i$  that acts on  $\downarrow_{\leq} x$ , implying  $g_i g_i^{-1} \in \mathbb{G}_{\downarrow \leq x}$ .
3. If  $y \notin \mathcal{Irrred}_{\leq}^{\vee} y$  there exist a unique irreducible  $\hat{y} \in T$  and an element  $\hat{x} \in T$  such that  $\hat{y} \in (\mathcal{Irrred}_{\leq}^{\vee} y \setminus \mathcal{Irrred}_{\leq}^{\vee} x)^{\mathbb{G}}$ ,  $\hat{x} \prec' \hat{y}$  and  $\hat{x} < x$  hold. Obviously  $\hat{y} \not\leq x$ . In that case we define  $\lambda_1(x, y) := \lambda_1(\hat{x}, \hat{y})$ .

Induction over the height (the size of the longest chain) leads to the desired annotation. Obviously we don't need to define any  $\lambda_2$  to something different than the empty set. Thus we can define  $\lambda := \lambda_1$  which fulfils the conditions of Definition 4 and provides a labelling using upper minimal automorphism.  $\square$

**Definition 6.** *Let  $\mathbb{V} = (V, \leq)$  a finite lattice and  $\mathbb{G} \leq \mathcal{A}ut \mathbb{V}$  a group of automorphisms. Let further  $T \subseteq V$  a transversal of the orbit partition  $V \parallel \mathbb{G}$  and  $\lambda : T \times T \rightarrow \mathfrak{P} \mathbb{G}$  a minimal acting annotation. Let further  $\leq'$  defined by  $x \leq' y$  iff there exists an automorphism  $g \in G$  such that  $x \leq y^g$ . Then the triplet  $(T, \leq', \lambda)$  is called minimal acting orbifold of  $\mathbb{V}$  by  $\mathbb{G}$ .*



**Theorem 1 (Unfolding).** *Let  $\mathbb{V} = (V, \leq)$  a finite lattice and  $\mathbb{G} \leq \mathfrak{Aut} \mathbb{V}$  a group of automorphisms and  $(T, \leq', \lambda)$  a hierarchical (minimal acting) orbifold of  $\mathbb{V}$  by  $\mathbb{G}$ . Let further for  $\mathfrak{C} : T \times T \rightarrow \mathfrak{P}T$  the mapping that assigns a pair  $x \leq' y$  to the set of all contiguous chains of  $\lambda$  from  $x$  to  $y$ , and to the empty set otherwise (i. e. if  $x \not\leq' y$ ).*

*Then the ordered set  $(L, \preceq)$  with the relation  $\preceq \subseteq L \times L$  defined by*

$$L := \bigcup \{x^{\Lambda(0,x)} \mid x \in T\}, \quad \text{and} \quad (6)$$

$$x \preceq y :\Leftrightarrow \exists z, \hat{z} \in T, g \in \Lambda(0, \hat{z}) : z^g = x, \hat{z}^g = y, z \leq' \hat{z} \quad (7)$$

*equals  $(V, \leq)$ .*

*Proof.* We prove this theorem by induction. As any finite lattice is also a complete lattice the orbit of the minimal element 0 of  $(V, \leq)$  is a singleton. That implies that  $0 \in T$ .

Let us start with the set  $L_0 := \{0\}$  containing the infimum of the lattice and the relation  $\preceq_0 := \{(0, 0)\}$ .

Let  $y \in V$  and suppose that for any  $x < y$  we have already proved that  $\downarrow_{\preceq} x = \downarrow_{\leq} x \subseteq V$ . Thus, for each  $x \in \{x' \in V \mid x' \prec y\}$  there exists an automorphism  $g_x \in G$  such that  $x^{g_x} \in T$ . W.l.o.g.  $g_x \in \Lambda(0, x^{g_x})$  (otherwise there exists  $g' \in \Lambda(0, x^{g_x})$  with  $x^{g'} = y^{g_x}$ ). As  $T$  is a transversal of  $V$ , there is also an automorphism  $g_y \in G$  such that  $y^{g_y} \in T$ . From  $x \leq y$  we know  $x^{g_x} \leq' y^{g_y}$  and thus, if  $g_y \in \lambda(x^{g_x}, y^{g_y}) \cdot \Lambda(0, x^{g_x})$  then also  $x \preceq y$ .

Note that for any  $z$  the equation  $\Lambda(0, z) = \bigcup_{\hat{z} \prec' z} (\lambda(\hat{z}, z) \Lambda(0, \hat{z}))$  holds. If there exists an automorphism  $h \in \lambda(x^{g_x}, y^{g_y})$  such that  $y^{g_x} = (y^{g_y})^h$  then the condition  $h \cdot g_x^{-1} \in h \cdot \Lambda(0, x^{g_x}) \subseteq \lambda(x^{g_x}, y^{g_y}) \cdot \Lambda(0, x^{g_x})$  holds. Thus,  $y \in L$  and  $x \preceq y$ . If there is no such element in  $\lambda(x^{g_x}, y^{g_y})$ , then by Definition 4 we can find an automorphism  $h \in \lambda(x, y) \cdot \Lambda(0, x)$  which maps  $y^{g_y}$  to  $y^{g_x}$ . Thus,  $y \in L$  and  $x \preceq y$  hold in this case, too. As we had chosen  $x$  arbitrarily below  $y$  we have proved  $\downarrow_{\preceq} y \subseteq \downarrow_{\leq} y$ .

Since  $L$  is constructed by automorphisms of  $\mathbb{V}$  which map certain elements of  $V$  to other elements of  $V$ , we know  $L \subseteq V$ . Suppose that for any two elements  $x, y \in L$  the inequality  $x \preceq y$  holds. Then we know that in equation (7) the condition  $\lambda(z, \hat{z}) \cdot \Lambda(0, z) \subseteq \lambda_{full}(z, \hat{z})$  holds if  $\lambda_{full}$  is the full annotation as discussed in [1,2,3,4]. Thus, we know that for any automorphism  $g \in \lambda(z, \hat{z}) \cdot \Lambda(0, z)$  the inequality  $x = z^g \leq \hat{z}^g = y$  holds. Thus also  $x \leq y$ .

As we have proved the condition  $\downarrow_{\preceq} y = \downarrow_{\leq} y \subseteq V$ , induction proves the equation  $(L, \preceq) = (V, \leq)$  for  $y = 1 \in T$ .  $\square$

## 6 An Example with Musical Background

In many parts of computational music theory pitches and notes are represented by integers. This has been proved to be useful especially in technical applications. As there are well-documented mathematical models available (see e. g., [6,7,8,9]) and an applied description is available in [10], here only the technically necessary

parts are described. Let  $\mathbb{Z}$  be considered as tone system. Then each subset  $C \subseteq \mathbb{Z}$  can be considered as a chord. In music theory it is not very common to talk about tones. It is more common to talk about scales that consist of chromas. Let  $o \in \mathbb{Z}$  be an interval which we will call octave. Two tones which are an octave apart are considered to have the same chroma. The transitive continuation of this procedure leads to a structure of chromas that is isomorphic to  $\mathbb{Z}_o$ . Each of its subsets is called harmony. For certain applications (e.g. in the software “MUTABOR” [11]) the form of harmonies of incoming streams of music (e.g. a MIDI stream [12]) are of special interest. Two harmonies have the same form if there exists a transposition that transforms one into the other.

Let  $H \subseteq \mathbb{Z}_o$  a harmony. Then for some chromatic interval  $i \in \mathbb{Z}_o$  the mapping  $t_i : \mathfrak{P}\mathbb{Z}_o \rightarrow \mathfrak{P}\mathbb{Z}_o : H \mapsto \{p + i \mid p \in H\}$  is called a transposition. The harmonic form  $F(H)$  of some harmony  $H$  is defined as the mapping  $F : \mathfrak{P}\mathbb{Z}_o \rightarrow \mathfrak{P}(\mathfrak{P}\mathbb{Z}_o) : H \mapsto \{t_i(H) \mid i \in \mathbb{Z}_o\}$ .

As for each interval  $i \in \mathbb{Z}_o$  there exists a transposition  $t_i$ . These transpositions can be considered as automorphisms of the ordered set  $(\mathfrak{P}\mathbb{Z}_o, \subseteq)$ , the transposition group will be denoted by  $\mathbb{T}$ . In fact this ordered set is a complete lattice which is invariant under transposition. The harmonic forms can be considered as the set of the orbits of the transpositions  $\mathfrak{P}\mathbb{Z}_o \parallel \mathbb{T}$ .

If we want to recognise a certain set of harmonies  $\mathfrak{H}$  we can build an automaton that can be described by a concept lattice. Let  $\mathbb{H} = \mathbb{K}(G, M, I)$  be the context defined by

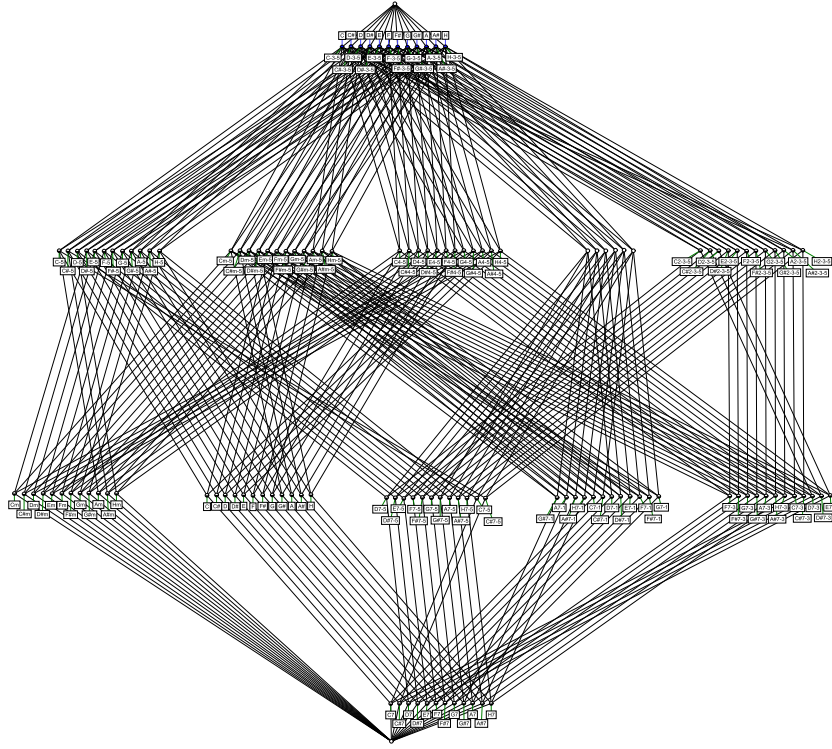
$$G := \bigcup_{H \in \mathfrak{H}} \mathfrak{P}H, \quad M := \mathbb{Z}_o, \quad \text{and} \quad I := \{(H, p) \in G \times M \mid p \in H\}. \quad (8)$$

Then  $\mathfrak{B}\mathbb{H}$  can be considered as automaton that recognises all finite words that consist of letters which are included in one of the harmonies of  $\mathfrak{H}$ . Starting in the concept  $(G, \emptyset)$ , with each pitch  $p \in \mathbb{Z}_o$  the automaton switches state  $(A, B)$  to state  $((B \cup \{p\})^I, B \cup \{p\})$ . The latter is a state as with every Harmony  $H$  the set of objects  $G$  includes each of its subsets  $H' \subseteq H$ . If such a state doesn't exist the automaton won't recognise the word.

The naive approach to recognise harmonic forms uses the same idea. Let  $\mathfrak{F} := \{F(H) \mid H \in \mathfrak{H}\}$  a set of harmonic forms. Then we define the lattice as follows:  $\mathbb{F} = \mathbb{K}(G', M', I')$  with

$$G := \{t_i(H), H \in \mathfrak{H}, i \in \mathbb{Z}_o\}, \quad M := \mathbb{Z}_o, \quad \text{and} \quad I := \{(H, p) \in G \times M \mid p \in H\}. \quad (9)$$

The concept lattice  $\mathfrak{B}(\mathbb{F})$  has all transpositions as automorphisms. Figure 6 shows a concept lattice that can be used to recognise the major seventh chord  $F(\{0, 4, 7, 10\})$ , the minor triad  $F(\{0, 3, 7\})$  and all of their harmonic subforms. The nodes are arranged orbit-wise. That means, each cluster is an orbit of  $\mathfrak{B}(\mathbb{F}) \parallel \mathbb{T}$ . Thus, the automorphisms can be seen as cyclic permutations of the endpoints of the edges. In comparison with the number of orbits the lattice is large: 14 orbits are formed by 140 concepts.



**Figure 2.** Concept lattice describing patterns to be matched

For an automaton that recognises harmonic forms it would be interesting to compress the data as the generation of the lattice can be very time and space consuming if the chroma system contains more chromas. E. g. considering the pitch bend parameter as part of a pitch in standard MIDI environments the number of pitches increases from 12 to  $12 \cdot 2^{14}$ . In such a case an orbifold based representation of the lattice does not necessarily increase in size. Starting by a hierarchical annotation of minimal acting automorphisms we can enhance the annotation by replacing each automorphism by a pair consisting of the automorphism and the character (pitch) that triggers its action. To avoid unnecessary operations the automaton could save the automorphism that must be applied to the pattern rather than applying it. In many cases (e. g., classification) it does not need to be applied at all.

This approach provides two additional advantages: As we know the context automorphisms, we can use a folded context to compute the order relation of the concept orbifolds as described in [4]. On the other hand changing the size of the chroma system can be done in several ways. The orbifold based approach provides a promising base for analysing such operations in order to provide fast algorithms that can be used in real time.

## 7 Outlook

We have seen that orbifolds of certain lattices can be described using hierarchical annotations, and that it is possible to minimise the action of the annotating automorphisms without losing the possibility of unfolding such hierarchical orbifolds.

Nevertheless there are open topics that can improve the theory. In Lemma 6 Restriction (5) has technical reasons. At the moment it is an open question how to deal with arbitrary lattices. It might be helpful to use systems of generators for the annotation  $\lambda$ . That should be straight forward if care is taken on conjugated subgroups.

Another easy extension would be to elaborate the idea for arbitrary ordered sets.

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