The lattice of all betweenness relations : Structure and properties

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Abstract. We consider implication bases with premises of size exactly 2, which are also known as *betweenness* relations. Our motivations is that several problems in graph theory can be modelled using betweenness relations, e.g. hull number, maximal cliques. In this paper we characterize the lattice of all betweenness relations by giving its poset of *irreducible* elements. Moreover, we show that this lattice is a meet-sublattice of the lattice of all *closure systems.*

1 Introduction

A *convexity space* on a ground set X is a subset of 2^X that is closed under intersection. Convexity spaces were studied in [13] and are sometimes called *Closure systems.* The members of a convexity space are called *convex sets.* Since the paper [13], convexity spaces are studied by several authors who describe several of their properties (see the joint paper [8] of Edelman and Jamison for a list of publications during the eighties), in particular the set of convexity spaces forms a *lattice.*

In this paper we deal with *betweenness* relations which are special cases of convexity spaces. The notion of betweenness relation has appeared in the early twentieth century when mathematicians focused on fundamental geometry [4]. A *betweenness* relation B on a finite set X is a set of triples $(x, y, z) \in X^3$. The most intuitive betweenness relations are those coming from metric spaces (a point y is between x and z if they satisfy the triangular equality). A *convex set* of a betweenness relation B is a subset Y of X such that for all $(x, y, z) \in B$, if $\{x, z\} \subseteq Y$, then $y \in Y$. It is well-known that the set of convex sets of a betweenness relation is a convexity space. Betweenness relations have been thoroughly studied by Menger and his students [16]. Other betweenness relations have arisen in research fields as probability theory with the work of Reichenbach [18]. Betweenness relations have been also studied in graphs in order to generalize geometrical theorems [1,2,9] (see the survey [17] for a non exhaustive list of betweenness relations on graphs).

We are interested in describing the set of all betweenness relations on a ground set X. We prove that the set of all convexity spaces on X derived from betweenness relations on X forms a lattice and is in fact a *meet-sublattice* of the lattice of all convexity spaces on X (we give an example showing that it is not

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a sub-lattice). We also describe the set of *meet* and *join-irreducible* elements of the lattice. We conclude by showing that the set of convexity spaces obtained from *clique betweenness* relations on graphs is a sublattice of the lattice of all convexity spaces of betweenness relations.

This paper is motivated by understanding links between several parameters that are considered in different areas such as FCA, database, logic and graph theory.

Summary. Notations and definitions are given in Section 2. The description of the lattice of convexity spaces of betweenness relations is given in Section 3. The convexity spaces of clique betweenness relations is described in Section 4. Some questions arising from algorithmic aspects are given in Section 5.

2 Preliminaries

Let X be finite set. A *partially ordered set* on X (or *poset*) is a reflexive, anti-symmetric and transitive binary relation denoted by $P := (X, \leq)$. For $x, y \in X$, we say that y covers x, denoted by $x \prec y$, if for any $z \in X$ with $x \leq z \leq y$ we have $x = z$ or $y = z$. A *lattice* $L := (X, \leq)$ is a partially ordered set with the following properties:

1. for all $x, y \in X$ there exists a unique z, denoted by $x \vee y$, such that for all $t \in X$, $t \geq x$ and $t \geq y$ implies $z \leq t$. (Upper bound property.)
2. for all $x, y \in X$ there exists a unique z, denoted by $x \wedge y$, such that for all $t \in X$, $t \leq x$ and $t \leq y$ implies $z \geq t$. (Lower bound property.)

Let $L = (X, \leq)$ be a lattice. An element $x \in X$ is called *join-irreducible* (resp. *meet-irreducible*) if $x = y \vee z$ (resp. $x = y \wedge z$) implies $x = y$ or $x = z$. A join-irreducible (resp. meet-irreducible) element covers (resp. is covered by) exactly one element. We denote by J_L and M_L the set of all join-irreducible and meet-irreducible elements of L respectively.

The poset of irreducible elements of a lattice $L = (X, \leq)$ is a representation of L by a bipartite poset $Bip(L) = (J_L, M_L, \leq)$. The concept lattice of $Bip(L)$ is isomorphic to L (for more details see the books of Davey and Priestley [3], and Ganter and Wille [10]).

An implication on X is an ordered pair (A, B) of subsets of X, denoted by $A \to B$. The set A is called the premise and the set B the conclusion of the implication $A \to B$. Let Σ be a set of implications on X. A subset $Y \subseteq X$ is Σ-closed if for each implication $A \to B$ in Σ, $A \subseteq Y$ implies $B \subseteq Y$. The closure of a set S by Σ, denoted by S^{Σ}, is the smallest Σ-closed set containing S.

Let S be a subset of X. Algorithm 1 computes the closure of S by a betweenness relation Σ. It is known as *forward chaining procedure* or *chase procedure* [11].

The set of Σ-closed subsets of X, denoted by F_{Σ}, is a *closure system* on X (i.e closed under set-intersection), and when ordered under inclusion is a lattice.

Algorithm 1: Set Closure(S, Σ)

Data: A set $S \subseteq X$ and Σ a betweenness
Result: The closure S^{Σ}
begin
  Let $S^{\Sigma} := S$;
  while $\exists xy \to z \in \Sigma$ *s.t.* $\{x, y\} \subseteq S^{\Sigma}$ *and* $z \notin S^{\Sigma}$ **do**
    $S^{\Sigma} = S^{\Sigma} \cup \{z\}$;
end

Conversely, given a closure system $\mathcal{F}$ on X, a family Σ of implications on X is called an implicational basis for $\mathcal{F}$ if $\mathcal{F} = F_{\Sigma}$. A subset $K \in X$ is called a *key* if $K^{\Sigma} = X$ and K is minimal under inclusion with this property. The name key comes from database theory [15].

Definition 1. *An implication set Σ on X is called a* betweenness relation *if for all $A \to B \in \Sigma$, $|A| = 2$.*

Two betweenness relations Σ_1 and Σ_2 are said to be *equivalent*, denoted by $\Sigma_1 \equiv \Sigma_2$, if $F_{\Sigma_1} = F_{\Sigma_2}$. We define the *closure* of a betweenness relation Σ by $\Sigma^c = \{ab \to c \mid a, b, c \in X \text{ and } \Sigma \equiv \Sigma \cup \{ab \to c\}\}$. Note that Σ^c is the unique maximal betweenness relation equivalent to Σ. In each equivalence class we distinguish two types of betweenness relations:

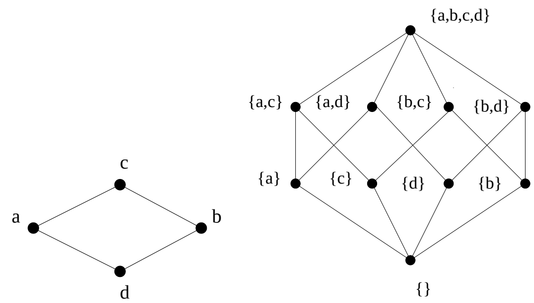
Canonical A *canonical betweenness* is the maximum in its equivalence class.
Optimal A betweenness Σ is *optimal* if for any betweenness relation Σ' equivalent to Σ, we have $|\Sigma| \leq |\Sigma'|$.

A graph G is a pair $(V(G), E(G))$, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges. We consider simple graphs (for further definitions see the book [6]). Examples of betweenness relations arising from graph theory are $\Sigma_G = \{xy \to z \mid z \text{ lies in a shortest path from } x \text{ to y}\}$ and $\Sigma_G = \{xy \to V(G) \mid xy \in E(G)\}$. Several other notions of convexity spaces are defined on graphs (see [17]). Figure 1 gives an example of a graph and its convex sets for the shortest path betweenness.

3 The Lattice of all Betweenness Relations

Let X be a finite set and Σ a betweenness relation on X. Given two sets A and C in 2^X such that $A \subseteq C$, we define the *set interval* $[A, C]$ as the family of all sets B in 2^X such that $A \subseteq B \subseteq C$.

Demetrovics *et al.* [5] gave a characterization of convex sets of an implication basis. Proposition 1 is restricted to betweenness relations.



(a) A graph G (b) Convex sets of G for shortest path betweenness

Fig. 1. A graph and its convex sets for the shortest path betweenness relation

Proposition 1. *[5] Let Σ be a betweenness relation on a set X. Then,*

$$F_\Sigma = 2^X \setminus \bigcup_{ab \to c \in \Sigma} [\{a,b\}, X \setminus \{c\}].$$

We denote by $\mathbb{F}_X := \{F_\Sigma \mid \Sigma \text{ is a betweenness relation on } X\}$ the family of all betweenness relations on X.

Theorem 1. *$\mathbb{F}_X$ is a closure system and therefore a lattice when structured under inclusion.*

Proof. We have to prove that this structure is closed under the intersection and it contains a unique maximal element.

Let $F_1, F_2 \in \mathbb{F}_X$, then there exist Σ_1 and Σ_2 inducing these families of convex sets on X. Let $F = F_1 \cap F_2$ and Σ be the betweenness defined by $ab \to c \in \Sigma$ if $ab \to c \in \Sigma_1$ or $ab \to c \in \Sigma_2$. Then we claim that $F = F_\Sigma$.

Let C be a set in F. It is convex for Σ_1 and for Σ_2. Therefore, for any a, b in C, every c such that $ab \to c \in \Sigma_1$ or $ab \to c \in \Sigma_2$ is in C. From this, we derive that for any a, b in C, every c such that $ab \to c \in \Sigma$ is in C, so that C is convex for Σ.

Reciprocally, let C be a convex set for Σ. We will show that it is convex for Σ_1 and Σ_2. Let a, b, c be elements of X such that a and b are in C and $ab \to c \in \Sigma_1$. Then $ab \to c \in \Sigma$ and since C is convex for Σ, b is in C. Therefore, C is in F_1. Similarly it is in F_2 so that it is in F.

Since $\Sigma = \Sigma_1 \cup \Sigma_2$, we conclude that Σ is a betweenness relation and therefore the set $\mathbb{F}_X$ is closed under intersection.

The family 2^X is in $\mathbb{F}_X$. It arises when Σ is the empty betweenness relation.

Proposition 2. *Given two families F_1 and F_2 in $\mathbb{F}_X$ and their canonical betweenness relations Σ_1 and Σ_2. Then*

– $F_1 \wedge F_2 = F_{\Sigma_1 \cup \Sigma_2}$.
– $F_1 \vee F_2 = F_{\Sigma_1 \cap \Sigma_2}$.

Proof. See the proof of Theorem 1 for the first point. Noww, note that Σ_1 and Σ_2 are canonical betweenness relations. Suppose that $F_1 \vee F_2$ is not $F_{\Sigma_1 \cap \Sigma_2}$. By Proposition 1, we have $F_1 \vee F_2 \subseteq F_{\Sigma_1 \cap \Sigma_2}$.

Now call $\Sigma_\vee$ the canonical betweenness relation related to $F_1 \vee F_2$. If $ab \to c \in \Sigma_\vee$ for some a, b and c in X, then we know that $ab \to c \in \Sigma_i$ because $F_i \subseteq F_1 \vee F_2$ for $i = 1, 2$ (in the canonical form, we have every $ab \to c$ such that the corresponding interval has no intersection with F). Therefore, every $ab \to c \in \Sigma_\vee$ is true for Σ, and $F_{\Sigma_1 \cap \Sigma_2} = F_1 \vee F_2$.

Corollary 1. *$\mathbb{F}_X$ is a meet-sublattice of the lattice of all closure systems on X.*

Proof. Let F_1 and F_2 be two families in $\mathbb{F}_X$. Since $F_1 \cap F_2$ is the closure system of a betweenness relation then the meet is preserved and thus $\mathbb{F}_X$ is a a meet-sublattice of the lattice of all closure systems on X.

Remark 1. Notice that $\mathbb{F}_X$ is not a sublattice of the lattice of all closure systems on X. It suffices to consider the example where $X = \{1, 2, 3, 4\}$. Take the co-atoms $F_1 = 2^X \setminus [\{1, 2\}, \{1, 2, 3\}]$ defined by the betweenness relation restricted to $12 \to 4$ and $F_2 = 2^X \setminus [\{2, 3\}, \{1, 2, 3\}]$ defined by the betweenness relation restricted to $23 \to 4$. Then $F_1 \cup F_2 = 2^X \setminus \{\{1, 2, 3\}\}$ and is a closure system, while $F_1 \vee F_2$ is the top element, 2^X.

In the following, we give a characterization of the poset of irreducible elements of $\mathbb{F}_X$.

Proposition 3. *The poset of irreducible elements of $\mathbb{F}_X$ is the bipartite poset $Bip(\mathbb{F}_X) = (J_{\mathbb{F}_X}, M_{\mathbb{F}_X}, \subseteq)$ where*

$$J_{\mathbb{F}_X} := \{F_\perp \cup \{S\} \mid S \in 2^X \setminus F_\perp\} \text{ where } F_\perp = \{\emptyset, X\} \cup \{\{x\} \mid x \in X\}$$
$$M_{\mathbb{F}_X} := \{2^X \setminus [ab, X \setminus \{c\}] \mid a, b, c \in X\}.$$

Proof. We prove this proposition point by point.

Consider the maximal betweenness relation $\Sigma = \{ab \to c \mid a, b, c \in X\}$. Then $F_\perp = F_\Sigma = 2^X \setminus \bigcup_{ab \to c \in \Sigma} [\{a, b\}, X \setminus \{c\}]$ (see Proposition 1). Thus $F_\perp = \{\emptyset, X\} \cup \{\{x\} \mid x \in X\}$.

For meet-irreducible elements, we will first consider co-atoms. Let Σ be a betweenness relation such that F_Σ is a co-atom. Since Σ is non-empty, then there exists a set which is not convex. Thus Σ must contain an implication $ab \to c$, with $a, b, c \in X$, which corresponds to the maximal closure system in $\mathbb{F}_X$ and different from 2^X. Namely, we remove from 2^X the convex sets of the interval $[\{a, b\}, X \setminus \{c\}]$.

Now suppose there exists another meet-irreducible element F which is not a co-atom. Call Σ_1 a betweenness relation such that F is F_{Σ_1}. Also, call F' the only successor of F_{Σ_1} and Σ_2 a betweenness such that F' is F_{Σ_2}. By Proposition 1, we know that from F' to F, we remove at least an interval of the form $[\{a,b\}, X \setminus \{c\}]$. Thus, the co-atom associated to the implication $ab \rightarrow c$ is not above F' but it is above F, so that F has at least two successors. We conclude that all meet-irreducible elements are co-atoms of $\mathbb{F}_X$.

For join-irreducible elements, we will first characterize atoms of $\mathbb{F}_X$. Pick any subset S of X which is not empty, a singleton or the whole of X. Define the betweenness relation Σ_S such that $ab \rightarrow c \in \Sigma_S$ for every a, b, c in X except those where a and b are in S and c is not in S. Then F_{Σ_S} is $F_\perp \cup \{S\}$. Now suppose there exists a join-irreducible $F \in \mathbb{F}_X$ that is not an atom. F contains at least one set S which is not in the unique closure system $F' \in \mathbb{F}_X$ that it covers. But there is an atom which contains exactly $F" = F_\perp \cup \{S\}$, with $F" \subseteq F$ and $F" \not\subseteq F'$. Thus F covers at least two elements, and thus F is not a join-irreducible element.

Corollary 2. *$\mathbb{F}_X$ contains $\binom{n}{2}(n-2)2^{n-3}$ meet-irreducible and $2^n - (n+2)$ join-irreducible elements.*

Proof. Every co-atom is of the form $2^X \setminus [\{a,b\}, X \setminus \{c\}]$ and is above every atom formed by $F_\perp$ and any set not in the forbidden interval. This makes $2^n - (n+2) - 2^{n-3}$ atoms below it.

An atom of the form $F_\perp \cup S$ where S is a set on X of size 2 to $n-1$, is below every co-atom that does not forbid S. This number equals $\binom{n}{2}(n-2)2^{n-3} - [\binom{|S|}{2}(n-|S|)]$.

Figure 2 shows the irreducible poset for $X = \{1,2,3,4\}$ where every atom is represented by the set S added to $F_\perp$ and every co-atom is represented by the removed implication $xy \rightarrow z$.

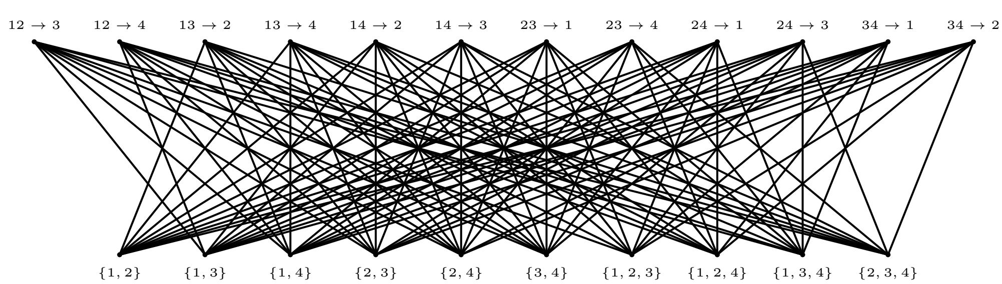


Fig. 2. Irreducible poset for $n = 4$

4 Clique Betweenness Relations

In this section we deal with a special betweenness relation defined through a graph in a specific way. Many other betweenness relations on graphs can be defined in the same way, e.g. independent sets, set covers.

Given a graph G, we define the betweenness relation

$$\Sigma_G := \{ab \to c \mid c \in X,\ ab \notin E(G)\}.$$

The convex sets of Σ_G are exactly the cliques of G. Notice that for any graph G, there is a corresponding betweenness relation Σ_G. In the following, we characterize the lattice of all Σ_G where G is a graph with vertex-set X. We denote by $\mathbb{F}_X^K = \{F_{\Sigma_G} \mid G \text{ is a graph on } X\}$.

Proposition 4. *$(\mathbb{F}_X^K, \subseteq)$ is a lattice.*

Proof. The bottom (resp. top) element of $\mathbb{F}_X$ corresponds to F_{Σ_G} where G is a stable (resp. clique) on X.

Moreover, $\mathbb{F}_X^K$ is closed under intersection (the cliques of the intersection of two graphs are exactly the ones in the intersection of both families of cliques). Therefore, $(\mathbb{F}_X^K, \subseteq)$ is a lattice.

Since Σ_G is a betweenness relation, then $\mathbb{F}_X^K \subset \mathbb{F}_X$ for any graph G defined on X.

Proposition 5. *The lattice $(\mathbb{F}_X^K, \subseteq)$ is a sublattice of $(\mathbb{F}_X, \subseteq)$.*

Proof. It is easy to see that it is a meet-sublattice, the meet of two families is the intersection of both families in both structures.

In order to prove that it is a join-sublattice of $(\mathbb{F}_X, \subseteq)$, consider two graphs G_1 and G_2 and their clique families F_1 and F_2. Call G the graph union of G_1 and G_2 and F the family of its cliques. The related betweenness relation is obtained by taking the intersection of betweenness relations related to G_1 and G_2 (non-edges in G are exactly non-edges in G_1 and in G_2). Therefore it is the join of F_1 and F_2 in $(\mathbb{F}_X, \subseteq)$.

Corollary 3. *The lattice $(\mathbb{F}_X^K, \subset)$ is a boolean lattice with $\binom{n}{2}$ atoms.*

Proof. For each graph G, its corresponding canonical betweenness relation is the set $\Sigma_G^c := \{ab \to X \mid ab \notin E(G)\}$. Note that any super-set of Σ_G^c corresponds to a betweenness relation of a partial graph of G, by deleting edges ab from G which corresponds to adding $ab \to X$ in Σ_G^c. Since any atom of $(\mathbb{F}_X^K, \subset)$ corresponds to a betweenness relation which contains exactly an implication $ab \to X$, we have exactly $\binom{n}{2}$ atoms.

5 Algorithmic Aspects of Betweenness Relations

In this section, we recall some optimization problems related to betweenness relations.

Minimum Key (MK)
Input: Σ a betweenness relation on X and k an integer.
Question: Is there a set $K \subseteq X$ such that $|K| \leq k$ and $K^{\Sigma} = X$?

These problems have been studied in the several domains and specially in database theory, and they have been proved NP-complete [5,14,15] for general implication bases. The problem MK has been proved NP-complete for particular cases of betweenness relations (see for instance [7] for the shortest path betweenness relation on graphs). It is known as the *hull number* of a betweenness relation. Therefore, we have the following.

Proposition 6. *MK is NP-complete.*

Recently, Kanté and Nourine [12] have shown that MK is polynomial for shortest path betweenness relations of chordal and distance hereditary graphs by using database techniques. Can we use the lattice structure of $\mathbb{F}_X$ to get new polynomial time algorithms for the MK problem in new graph classes?

Now we consider the problem which computes an optimal cover of a betweenness relation. This problem is to find an optimal betweenness relation which is equivalent to a given betweenness relation. Several works have been done in the general case known as *Horn minimization* [11].

Optimal Cover (OC)
Input: Σ a betweenness relation on X and k an integer.
Question: Is there a betweenness relation Σ' equivalent to Σ such that $|\Sigma'| \leq k$?

The size of an optimal cover is known as the *hydra number* [19]. The computational complexity of the hydra number is open for betweenness relations, but is NP-complete for the general case [11,15]. We hope that the lattice structure of $\mathbb{F}_X$ could help to address this question.

6 Conclusion

In this paper, we characterize the context of the lattice of all betweenness relations on a finite set, which is a meet-sublattice of the lattice of all closure systems on the same set. We are convinced that the structure of lattice can help to understand some problems of graph theory such as hull number and hydra number. In the future we will investigate the link of these parameters and the structure of the lattice.

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