Linking L-Chu correspondences and completely lattice L-ordered sets

Ondrej Krídlo[1] and Manuel Ojeda-Aciego[2]

[1] University of Pavol Jozef Šafárik, Košice, Slovakia*
[2] Dept. Matemática Aplicada, Univ. Málaga, Spain**

Abstract. Continuing our categorical study of L-fuzzy extensions of formal concept analysis, we provide a representation theorem for the category of L-Chu correspondences between L-formal contexts and prove that it is equivalent to the category of completely lattice L-ordered sets.

1 Introduction

This paper deals with an extremely general form of Formal Concept Analysis (FCA) based on categorical constructs and L-fuzzy sets. FCA has become an extremely useful theoretical and practical tool for formally describing structural and hierarchical properties of data with "object-attribute" character, and this applicability justifies the need of a deeper knowledge of its underlying mechanisms: and one important way to obtain this extra knowledge turns out to be via generalization and abstraction.

Several approaches have been presented for generalizing the framework and the scope of formal concept analysis and, nowadays, one can see works which extend the theory by using ideas from fuzzy set theory, rough set theory, or possibility theory [1, 10, 18, 20–22, 24].

Concerning applications of fuzzy formal concept analysis, one can see papers ranging from ontology merging [9], to applications to the Semantic Web by using the notion of concept similarity or rough sets [11, 12], and from noise control in document classification [19] to the development of recommender systems [7].

We are concerned in this work with the category L-ChuCors, built on top of several fuzzy extensions of the classical concept lattice, mainly introduced by Bělohlávek [3, 5, 6], who extended the underlying interpretation on classical logic to the more general framework of L-fuzzy logic [13].

The categorical treatment of morphisms as fundamental structural properties has been advocated by [17] as a means for the modelling of data translation, communication, and distributed computing, among other applications. Our approach broadly continues the research line which links the theory of Chu spaces with concept lattices [25] but, particularly, is based on the notion of Chu correspondences between formal contexts developed by Mori in [23]. Previous work

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in this categorical approach has been developed by the authors in [14, 16]. The category L-ChuCors is formed by considering the class of L-contexts as objects and the L-fuzzy Chu correspondences as arrows between objects. Recently, the authors developed a further abstraction [15] aiming at formally describing structural properties of intercontextual relationships of L-contexts.

The main result in this work is a constructive proof of the equivalence between the categories of L-formal contexts and L-Chu correspondences and that of completely lattice L-ordered sets and their corresponding morphisms. In order to obtain a reasonably self-contained document, Section 2 introduces the basic definitions concerning the L-fuzzy extension of formal concept analysis, as well as those concerning L-Chu correspondences; then, the categories associated to L-formal contexts and L-CLLOS are defined in Section 3 and, finally, the proof of equivalence is in Section 4.

2 Preliminaries

2.1 Basics of *L*-fuzzy FCA

Definition 1. *An algebra* $\langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ *is said to be a* ***complete residuated lattice*** *if*

- $\langle L, \wedge, \vee, 0, 1\rangle$ *is a complete lattice with the least element* 0 *and the greatest element* 1*,*
- $\langle L, \otimes, 1\rangle$ *is a commutative monoid,*
- $\otimes$ *and* $\to$ *are adjoint, i.e.* $a \otimes b \le c$ *if and only if* $a \le b \to c$*, for all* $a, b, c \in L$*, where* $\le$ *is the ordering in the lattice generated from* $\wedge$ *and* $\vee$*.*

Definition 2. *Let L be a complete residuated lattice, an* L***-fuzzy context*** *is a triple $\langle B, A, r\rangle$ consisting of a set of objects B, a set of attributes A and an L-fuzzy binary relation r, i.e. a mapping $r\colon B \times A \to L$, which can be alternatively understood as an L-fuzzy subset of $B \times A$.*

Definition 3. *Consider an L-fuzzy context $\langle B, A, r\rangle$. Mappings $\uparrow\colon L^B \to L^A$ and $\downarrow\colon L^A \to L^B$ can be defined for every $f \in L^B$ and $g \in L^A$ as follows:*

$$\uparrow(f)(a) = \bigwedge_{o \in B} \big(f(o) \to r(o,a)\big) \qquad\qquad \downarrow(g)(o) = \bigwedge_{a \in A} \big(g(a) \to r(o,a)\big)$$

Definition 4. *An* L***-fuzzy concept*** *is a pair $\langle f, g\rangle$ such that $\uparrow(f) = g$ and $\downarrow(g) = f$. The first component f is said to be the* ***extent*** *of the concept, whereas the second component g is the* ***intent*** *of the concept.*

The set of all L-fuzzy concepts associated to a fuzzy context $\langle B, A, r\rangle$ will be denoted as L-FCL(B, A, r).

An ordering between L-fuzzy concepts is defined as follows: $\langle f_1, g_1\rangle \le \langle f_2, g_2\rangle$ if and only if $f_1 \subseteq f_2$ ($f_1(o) \le f_2(o)$ for all $o \in B$) if and only if $g_1 \supseteq g_2$ ($g_1(o) \ge g_2(o)$ for all $a \in A$).

Theorem 1 (See [5]). *The poset* $(L\text{-FCL}(B, A, r), \leq)$ *is a complete lattice where*

$$\bigwedge_{j\in J} \langle f_j, g_j \rangle = \Big\langle \bigwedge_{j\in J} f_j, \uparrow (\bigwedge_{j\in J} f_j) \Big\rangle$$
$$\bigvee_{j\in J} \langle f_j, g_j \rangle = \Big\langle \downarrow (\bigwedge_{j\in J} g_j), \bigwedge_{j\in J} g_j \Big\rangle$$

Moreover a complete lattice $\mathcal{V} = \langle V, \leq \rangle$ *is isomorphic to* $L\text{-FCL}(B, A, r)$ *iff there are mappings* $\gamma : B \times L \to V$ *and* $\mu : A \times L \to V$*, such that* $\gamma(B \times L)$ *is* $\bigvee$*-dense and* $\mu A \times L$ *is* $\bigwedge$*-dense in* $\mathcal{V}$*, and* $(k \otimes l) \leq r(o, a)$ *is equivalent to* $\gamma(o, k) \leq \mu(a, l)$ *for all* $o \in B$*,* $a \in A$ *and* $k, l \in L$*.*

Bělohlávek has extended the fundamental theorem of concept lattices by Dedekind-MacNeille completion in fuzzy settings by using the notions of L-equality and L-ordering. All the definitions and related constructions given until the end of the section are from [6].

Definition 5. *A binary* L*-relation* $\approx$ *on* X *is called an* L***-equality*** *if it satisfies*

1. $(x \approx x) = 1$, *(reflexivity),*
2. $(x \approx y) = (y \approx x)$, *(symmetry),*
3. $(x \approx y) \otimes (y \approx z) \leq (x \approx z)$, *(transitivity),*
4. $(x \approx y) = 1$ *implies* $x = y$

L-equality is a natural generalization of the classical (bivalent) notion.

Definition 6. *An* L***-ordering*** *(or fuzzy ordering) on a set* X *endowed with an* L*-equality relation* $\approx$ *is a binary* L*-relation* $\preceq$ *which is compatible w.r.t.* $\approx$ *(i.e.* $f(x) \otimes (x \approx y) \leq f(y)$*, for all* $x, y \in X$*) and satisfies*

1. $x \preceq x = 1$, *(reflexivity),*
2. $(x \preceq y) \wedge (y \preceq x) \leq (x \approx y)$, *(antisymmetry),*
3. $(x \preceq y) \otimes (y \preceq z) \leq (x \preceq z)$, *(transitivity).*

If $\preceq$ *is an* L*-order on a set* X *with an* L*-equality* $\approx$*, we call the pair* $\langle \langle X, \approx \rangle \preceq \rangle$ *an* L*-ordered set.*

Clearly, if $L = 2$, the notion of L-order coincides with the usual notion of (partial) order.

Definition 7. *An* L*-set* $f \in L^X$ *is said to be an* L***-singleton*** *in* $\langle X, \approx \rangle$ *if it is compatible w.r.t.* $\approx$ *and the following holds:*

1. *there exists* $x_0 \in X$ *with* $f(x_0) = 1$
2. $f(x) \otimes f(y) \leq (x \approx y)$*, for all* $x, y \in X$*.*

Definition 8. *For an* L*-ordered set* $\langle \langle X, \approx \rangle \preceq \rangle$ *and* $f \in L^X$ *we define the* L*-sets* $\inf(f)$ *and* $\sup(f)$ *in* X *by*

1. $\inf(f)(x) = (\mathcal{L}(f))(x) \wedge (\mathcal{UL}(f))(x)$
2. $\sup(f)(x) = (\mathcal{U}(f))(x) \wedge (\mathcal{LU}(f))(x)$

where

- $\mathcal{L}(f)(x) = \bigwedge_{y \in X} \big(f(y) \to (x \preceq y)\big)$
- $\mathcal{U}(f)(x) = \bigwedge_{y \in X} \big(f(y) \to (y \preceq x)\big)$

$\inf(f)$ *and* $\sup(f)$ *are called* infimum *or* supremum*, respectively.*

Definition 9. *An L-ordered set $\langle\langle X, \approx\rangle \preceq\rangle$ is said to be* ***completely lattice L-ordered set*** *if for any $f \in L^X$ both $\sup(f)$ and $\inf(f)$ are $\approx$-singletons.*

By proving of all the following lemmas some of the properties of residuated lattices are used. All details could be found in [4]. Some of needed properties are listed below.

1. $(k \to (l \to m)) = ((k \otimes l) \to m) = ((l \otimes k) \to m) = (l \to (k \to m))$
2. $k \to \bigwedge_{i \in I} m_i = \bigwedge_{i \in I}(k \to m_i)$
3. $(\bigvee_{i \in I} m_i) \to k = \bigwedge_{i \in I}(m_i \to k)$

Lemma 1. *For any pair of L-concepts $\langle f_i, g_i\rangle \in L\text{-FCL}(B, A, r)$ ($i \in \{1, 2\}$) of any L-context $\langle B, A, r\rangle$ the following equality holds.*

$$\bigwedge_{o \in B} \big(f_1(o) \to f_2(o)\big) = \bigwedge_{a \in A} \big(g_2(a) \to g_1(a)\big)$$

Proof.

$$\begin{aligned}
\bigwedge_{o \in B} \big(f_1(o) \to f_2(o)\big) &= \bigwedge_{o \in B} \big(f_1(o) \to \downarrow(g_2)(o)\big) \\
&= \bigwedge_{o \in B} \Big(f_1(o) \to \bigwedge_{a \in A} \big(g_2(a) \to r(o, a)\big)\Big) \\
&= \bigwedge_{a \in A} \Big(g_2(a) \to \bigwedge_{o \in B} \big(f_1(o) \to r(o, a)\big)\Big) \\
&= \bigwedge_{a \in A} \big(g_2(a) \to \uparrow(f_1)(a)\big) \\
&= \bigwedge_{a \in A} \big(g_2(a) \to g_1(a)\big)
\end{aligned}$$ □

Definition 10. *We define an L-equality $\approx$ and L-ordering $\preceq$ on the set of formal concepts L-FCL(C) of L-context C as follows:*

1. $\langle f_1, g_1\rangle \preceq \langle f_2, g_2\rangle = \bigwedge_{o \in B} \big(f_1(o) \to f_2(o)\big) = \bigwedge_{a \in A} \big(g_2(a) \to g_1(a)\big)$
2. $\langle f_1, g_1\rangle \approx \langle f_2, g_2\rangle = \bigwedge_{o \in B} \big(f_1(o) \leftrightarrow f_2(o)\big) = \bigwedge_{a \in A} \big(g_2(a) \leftrightarrow g_1(a)\big)$

where $k \leftrightarrow m$ is defined as $(k \to m) \wedge (m \to k)$ for any $k, m \in L$.

Definition 11. *Let $C = \langle B, A, r\rangle$ be an L-fuzzy formal context and γ be an L-set from $L^{L\text{-FCL}(C)}$. We define L-sets of objects and attributes $\bigcup_B \gamma$ and $\bigcup_A \gamma$, respectively, as follows:*

1. $(\bigcup_B \gamma)(o) = \bigvee_{\langle f,g\rangle \in L\text{-FCL}(C)} (\gamma(\langle f, g\rangle) \otimes f(o))$, *for* $o \in B$
2. $(\bigcup_A \gamma)(a) = \bigvee_{\langle f,g\rangle \in L\text{-FCL}(C)} (\gamma(\langle f, g\rangle) \otimes g(a))$, *for* $a \in A$

Theorem 2. *Let $C = \langle B, A, r\rangle$ be an L-context. $\langle\langle L\text{-FCL}(C), \approx\rangle, \preceq\rangle$ is a completely lattice L-ordered set in which infima and suprema can be described as follows: for an L-set $\gamma \in L^{L-\text{FCL}(C)}$ we have:*

$$^1\inf(\gamma) = \Big\{\big\langle \downarrow(\bigcup_A \gamma), \uparrow\downarrow(\bigcup_A \gamma)\big\rangle\Big\} \qquad ^1\sup(\gamma) = \Big\{\big\langle \downarrow\uparrow(\bigcup_B \gamma), \uparrow(\bigcup_B \gamma)\big\rangle\Big\}$$

Moreover a completely lattice L-ordered set $\mathcal{V} = \langle\langle V, \approx\rangle, \preceq\rangle$ is isomorphic to $\langle\langle L\text{-FCL}(\langle B, A, r\rangle), \approx_1\rangle, \preceq_1\rangle$ iff there are mappings $\gamma : B \times L \to V$ and $\mu : A \times L \to V$, such that $\gamma(B \times L)$ is $\{0, 1\}$-supremum dense and $\mu(A \times L)$ is $\{0, 1\}$-infimum dense in $\mathcal{V}$, and $((k \otimes l) \to r(o, a)) = (\gamma(o, k) \preceq \mu(a, l))$ for all $o \in B$, $a \in A$ and $k, l \in L$. In particular, $\mathcal{V}$ is isomorphic to $\langle\langle L\text{-FCL}(V, V, \preceq), \approx_1\rangle, \preceq_1\rangle$.

2.2 *L*-Chu correspondences

Definition 12. *Consider two L-fuzzy contexts $C_i = \langle B_i, A_i, r_i\rangle, (i = 1, 2)$, then the pair $\varphi = (\varphi_L, \varphi_R)$ is called a **correspondence** from C_1 to C_2 if φ_L and φ_R are L-multifunctions, respectively, from B_1 to B_2 and from A_2 to A_1 (that is, $\varphi_L : B_1 \to L^{B_2}$ and $\varphi_R : A_2 \to L^{A_1}$).*

*The L-correspondence φ is said to be a **weak L-Chu correspondence** if the equality*

$$\bigwedge_{a_1 \in A_1} (\varphi_R(a_2)(a_1) \to r_1(o_1, a_1)) = \bigwedge_{o_2 \in B_2} (\varphi_L(o_1)(o_2) \to r_2(o_2, a_2)) \qquad (1)$$

holds for all $o_1 \in B_1$ and $a_2 \in A_2$.

*A weak Chu correspondence φ is an **L-Chu correspondence** if $\varphi_L(o_1)$ is an L-set of objects closed in C_2 and $\varphi_R(a_2)$ is an L-set of attributes closed in C_1 for all $o_1 \in B_1$ and $a_2 \in A_2$. We will denote the set of all L-Chu correspondences from C_1 to C_2 by L-ChuCors(C_1, C_2).*

Definition 13. *Given a mapping $\varpi : X \to L^Y$, we define $\varpi_+ : L^X \to L^Y$ for all $f \in L^X$ by $\varpi_+(f)(y) = \bigvee_{x \in X} \big(f(x) \otimes \varpi(x)(y)\big)$.*

3 Introducing the relevant categories

3.1 The category *L*-ChuCors

- **objects** L-fuzzy formal contexts

- **arrows** L-Chu correspondences
- **identity arrow** $\iota : C \to C$ of L-context $C = \langle B, A, r\rangle$
  - $\iota_L(o) = \downarrow\uparrow(\chi_o)$, for all $o \in B$
  - $\iota_R(a) = \uparrow\downarrow(\chi_a)$, for all $a \in A$
- **composition** $\varphi_2 \circ \varphi_1 : C_1 \to C_3$ **of arrows** $\varphi_1 : C_1 \to C_2$, $\varphi_2 : C_2 \to C_3$ ($C_i = \langle B_i, A_i, r_i\rangle$, $i \in \{1, 2\}$)
  - $(\varphi_2 \circ \varphi_1)_L : B_1 \to L^{B_3}$ defined as $(\varphi_2 \circ \varphi_1)_L(o_1) = \downarrow_3\uparrow_3 \big(\varphi_{2L+}(\varphi_{1L}(o_1))\big)$ where

$$\varphi_{2L+}(\varphi_{1L}(o_1))(o_3) = \bigvee_{o_2 \in B_2} \varphi_{1L}(o_1)(o_2) \otimes \varphi_{2L}(o_2)(o_3)$$

  - and $(\varphi_2 \circ \varphi_1)_R : A_3 \to L^{A_1}$ defined as $(\varphi_2 \circ \varphi_1)_R(a_3) = \uparrow_1\downarrow_1 \big(\varphi_{1R+}(\varphi_{2R}(a_3))\big)$ where

$$\varphi_{1R+}(\varphi_{2R}(a_3))(a_1) = \bigvee_{a_2 \in A_2} \varphi_{2R}(a_3)(a_2) \otimes \varphi_{1R}(a_2)(a_1)$$

All details about definition of the category L-ChuCors could be found in [15].

3.2 Category *L*-CLLOS

Here we define another category

Objects are completely lattice L-ordered sets (in short, L-CLLOS) i.e. our objects will be represented as $\mathcal{V} = \langle\langle V, \approx\rangle, \preceq\rangle$

Arrows are pairs of mappings between two L- CLLOSs i.e. $\langle s, z\rangle$ between $\mathcal{V}_1$ and $\mathcal{V}_2$, such that:
1. $s : V_1 \to V_2$,
2. $z : V_2 \to V_1$,
3. $(s(v_1) \preceq_2 v_2) = (v_1 \preceq_1 z(v_2))$, for all $(v_1, v_2) \in V_1 \times V_2$.

Identity arrow of $\langle\langle V, \approx\rangle, \preceq\rangle$ is a pair of identity morphisms $\langle id_V, id_V\rangle$

Composition of arrows is based on composition of mappings: consider two arrows $\langle s_i, z_i\rangle : \mathcal{V}_i \to \mathcal{V}_{i+1}$, where $i \in \{1, 2\}$. Composition is defined as follows:

$$\langle s_2, z_2\rangle \circ \langle s_1, z_1\rangle = \langle s_2 \circ s_1, z_1 \circ z_2\rangle.$$

Thus, given a pair of two arbitrary elements $(v_1, v_3) \in V_1 \times V_3$ then:

$$\begin{aligned}\big((s_2 \circ s_1)(v_1) \preceq_3 v_3\big) &= \big(s_2(s_1(v_1)) \preceq_3 v_3\big)\\ &= \big(s_1(v_1) \preceq_2 z_2(v_3)\big)\\ &= \big(v_1 \preceq_1 z_1(z_2(v_3))\big)\\ &= \big(v_1 \preceq_1 (z_1 \circ z_2)(v_3)\big)\end{aligned}$$

Associativity of composition follows trivially because of the associativity of composition of mappings between sets.

4 The categories L-ChuCors and L-CLLOS are equivalent

In this section we start to build the equivalence by introducing a functor Γ from L-ChuCors to L-CLLOS in the following way:

1. $\Gamma(C) = \langle\langle L\text{-FCL}(C), \approx\rangle, \preceq\rangle$ for any L-context C will be its L-concept L-CLLOS.
2. $\Gamma(\varphi) = \langle \varphi_\vee, \varphi_\wedge \rangle$. To any L-Chu correspondence $\varphi \in L\text{-ChuCors}(C_1, C_2)$, $\Gamma(\varphi)$ will be a pair of mappings $\langle \varphi_\vee, \varphi_\wedge \rangle$ defined as follows:
   - $\varphi_\vee\big(\langle f_1, g_1\rangle\big) = \Big\langle \downarrow_2\uparrow_2 \big(\varphi_{L+}(f_1)\big), \uparrow_2 \big(\varphi_{L+}(f_1)\big)\Big\rangle$
   - $\varphi_\wedge\big(\langle f_2, g_2\rangle\big) = \Big\langle \downarrow_1 \big(\varphi_{R+}(g_2)\big), \uparrow_1\downarrow_1 \big(\varphi_{R+}(g_2)\big)\Big\rangle$

   where $\langle f_i, g_i\rangle \in L\text{-FCL}(C_i)$ for $i \in \{1, 2\}$.

Lemma 2. *$\Gamma(\varphi) \in L\text{-CLLOS}(\Gamma(C_1), \Gamma(C_2))$ for any $\varphi \in L\text{-ChuCors}(C_1, C_2)$.*

Proof. Consider two arbitrary L-concepts $\langle f_i, g_i\rangle$ of $\langle\langle L\text{-FCL}(C_i), \approx_i\rangle, \preceq_i\rangle$ for $i \in \{1, 2\}$, such that $C_i = \langle B_i, A_i, r_i\rangle$.

$$\begin{aligned}
\varphi_\vee\big(\langle f_1, g_1\rangle\big) \preceq_2 \langle f_2, g_2\rangle
&= \Big\langle \downarrow_2\uparrow_2 \big(\varphi_{L+}(f_1)\big), \uparrow_2 \big(\varphi_{L+}(f_1)\big)\Big\rangle \preceq_2 \langle f_2, g_2\rangle \\
&= \bigwedge_{a_2\in A_2} \Big(g_2(a_2) \to \uparrow_2 \big(\varphi_{L+}(f_1)\big)(a_2)\Big) \\
&= \bigwedge_{a_2\in A_2} \Big(g_2(a_2) \to \bigwedge_{o_2\in B_2} \Big(\bigvee_{o_1\in B_1} \big(\varphi_L(o_1)(o_2) \otimes f_1(o_1)\big) \to r_2(o_2, a_2)\Big)\Big) \\
&= \bigwedge_{a_2\in A_2} \bigwedge_{o_1\in B_1} \Big(g_2(a_2) \to \Big(f_1(o_1) \to \bigwedge_{o_2\in B_2} \big(\varphi_L(o_1)(o_2) \to r_2(o_2, a_2)\big)\Big)\Big) \\
&= \bigwedge_{a_2\in A_2} \bigwedge_{o_1\in B_1} \Big(f_1(o_1) \to \Big(g_2(a_2) \to \bigwedge_{a_1\in A_1} \big(\varphi_R(a_2)(a_1) \to r_1(o_1, a_1)\big)\Big)\Big) \\
&= \bigwedge_{o_1\in B_1} \Big(f_1(o_1) \to \bigwedge_{a_1\in A_1} \Big(\bigvee_{a_2\in A_2} \big(\varphi_R(a_2)(a_1) \otimes g_2(a_2)\big) \to r_1(o_1, a_1)\Big)\Big) \\
&= \bigwedge_{o_1\in B_1} \Big(f_1(o_1) \to \downarrow_1 \big(\varphi_{R+}(g_2)\big)(o_1)\Big) \\
&= \langle f_1, g_1\rangle \preceq_1 \Big\langle \downarrow_1 \big(\varphi_{R+}(g_2)\big), \uparrow_1\downarrow_1 \big(\varphi_{R+}(g_2)\big)\Big\rangle \\
&= \langle f_1, g_1\rangle \preceq_1 \varphi_\wedge\big(\langle f_2, g_2\rangle\big)
\end{aligned}$$

□

Lemma 3. *For the identity arrow $\iota \in L\text{-ChuCors}(C, C)$ of any L-context $C = \langle B, A, r\rangle$, $\Gamma(\iota)$ is the identity arrow from $L\text{-CLLOS}(\Gamma(C), \Gamma(C))$.*

Proof. Consider any L-concept $\langle f, g\rangle$ from $L\text{-FCL}(C)$.

$$\begin{aligned}
\uparrow \big(\iota_{L+}(f)\big)(a) &= \bigwedge_{o\in B} \big(\iota_{L+}(f)(o) \to r(o, a)\big) \\
&= \bigwedge_{o\in B} \Big(\bigvee_{b\in B} \big(\iota_L(b)(o) \otimes f(b)\big) \to r(o, a)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \bigwedge_{o\in B}\bigwedge_{b\in B}\big(\big(\iota_L(b)(o)\otimes f(b)\big)\to r(o,a)\big)\\
&= \bigwedge_{o\in B}\big(f(b)\to \bigwedge_{b\in B}\big(\iota_L(b)(o)\to r(o,a)\big)\\
&= \bigwedge_{b\in B}\big(f(b)\to \uparrow\downarrow\uparrow\big(\chi_b\big)(a)\big)\\
&= \bigwedge_{b\in B}\big(f(b)\to r(b,a)\big) = \uparrow(f)(a)
\end{aligned}
$$

Therefore, we have $\iota_\vee(\langle f,g\rangle)=\langle f,g\rangle$. The proof for $\iota_\wedge$ is similar. □

Lemma 4. *Consider arbitrary $\varphi_i \in L\text{-ChuCors}(C_i, C_{i+1})$ for $i\in\{1,2\}$ and any element $o_1\in B_1$ and $g_3\in L^{A_3}$. Then*

$$\downarrow_1\big(\varphi_{1R^+}\big(\varphi_{2R^+}(g_3)\big)\big)(o_1)=\downarrow_1\big(\varphi_{1R^+}\big(\uparrow_2\downarrow_2\big(\varphi_{2R^+}(g_3)\big)\big)\big)(o_1).$$

Proof.

$$
\begin{aligned}
&\downarrow_1\big(\varphi_{1R^+}\big(\varphi_{2R^+}(g_3)\big)\big)(o_1)\\
&= \bigwedge_{a_1\in A_1}\big(\varphi_{1R^+}\big(\varphi_{2R^+}(g_3)\big)(a_1)\to r_1(o_1,a_1)\big)\\
&= \bigwedge_{a_1\in A_1}\big(\bigvee_{a_2\in A_2}\big(\varphi_{1R}(a_2)(a_1)\otimes\varphi_{2R^+}(g_3)(a_2)\big)\to r_1(o_1,a_1)\big)\\
&= \bigwedge_{a_2\in A_2}\big(\varphi_{2R^+}(g_3)(a_2)\to \bigwedge_{a_1\in A_1}\big(\varphi_{1R}(a_2)(a_1)\to r_1(o_1,a_1)\big)\big)\\
&= \bigwedge_{a_2\in A_2}\big(\varphi_{2R^+}(g_3)(a_2)\to \bigwedge_{o_2\in B_2}\big(\varphi_{1L}(o_1)(o_2)\to r_2(o_2,a_2)\big)\big)\\
&= \bigwedge_{o_2\in B_2}\big(\varphi_{1L}(o_1)(o_2)\to \bigwedge_{a_2\in A_2}\big(\varphi_{2R^+}(g_3)(a_2)\to r_2(o_2,a_2)\big)\big)\\
&= \bigwedge_{o_2\in B_2}\big(\varphi_{1L}(o_1)(o_2)\to \downarrow_2\uparrow_2\downarrow_2\big(\varphi_{2R^+}(g_3)\big)(o_2)\big)
\end{aligned}
$$

by applying the same chain of modifications in opposite way we will obtain

$$= \downarrow_1\big(\varphi_{1R^+}\big(\uparrow_2\downarrow_2\big(\varphi_{2R^+}(g_3)\big)\big)\big)(o_1)$$

□

Lemma 5. *Mapping Γ is closed under arrow composition.*

Proof. Consider $\varphi_i\in L\text{-ChuCors}(C_i,C_{i+1})$ for $i\in\{1,2\}$. Let $\langle f_i,g_i\rangle\in L\text{-FCL}(C_i)$ be an arbitrary L-context for all $i\in\{1,3\}$. Recall that

1. $\Gamma(\varphi_2\circ\varphi_1)=\big\langle(\varphi_2\circ\varphi_1)_\vee,(\varphi_2\circ\varphi_1)_\wedge\big\rangle$
2. $\Gamma(\varphi_2)\circ\Gamma(\varphi_1)=\big\langle\varphi_{2\vee}\circ\varphi_{1\vee},\varphi_{1\wedge}\circ\varphi_{2\wedge}\big\rangle$

The proof will be based on equality of corresponding elements of the previous pairs: only one part will be proved, the other one is similar.

$$
\begin{aligned}
&(\varphi_{1\wedge} \circ \varphi_{2\wedge})\big(\langle f_3, g_3\rangle\big) = \varphi_{1\wedge}\big(\varphi_{2\wedge}\big(\langle f_3, g_3\rangle\big)\big)\\
=&\Big\langle \downarrow_1 (\varphi_{1R^+}(\uparrow_2\downarrow_2 (\varphi_{2R^+}(g_3)))), \uparrow_1\downarrow_1 (\varphi_{1R^+}(\uparrow_2\downarrow_2 (\varphi_{2R^+}(g_3))))\Big\rangle\\
&\text{by lemma 4 we have}\\
=&\Big\langle \downarrow_1 (\varphi_{1R^+}(\varphi_{2R^+}(g_3))), \uparrow_1\downarrow_1 (\varphi_{1R^+}(\varphi_{2R^+}(g_3)))\Big\rangle = \star
\end{aligned}
$$

$$
\begin{aligned}
&\downarrow_1 (\varphi_{1R^+}(\varphi_{2R^+}(g_3)))(o_1) =\\
&= \bigwedge_{a_1\in A_1} \Big(\bigvee_{a_2\in A_2} (\varphi_{1R}(a_2)(a_1) \otimes \varphi_{2R^+}(g_3)(a_2)) \to r_1(o_1, a_1)\Big)\\
&= \bigwedge_{a_1\in A_1} \Big(\bigvee_{a_2\in A_2} (\varphi_{1R}(a_2)(a_1) \otimes \bigvee_{a_3\in A_3} (\varphi_{2R}(a_3)(a_2) \otimes g_3(a_3))) \to r_1(o_1, a_1)\Big)\\
&= \bigwedge_{a_1\in A_1} \Big(\bigvee_{a_3\in A_3} \big(\varphi_{1R^+}(\varphi_{2R}(a_3))(a_1) \otimes g_3(a_3)\big) \to r_1(o_1, a_1)\Big)\\
&= \bigwedge_{a_3\in A_3} \Big(g_3(a_3) \to \bigwedge_{a_1\in A_1} \big(\varphi_{1R^+}(\varphi_{2R}(a_3))(a_1) \to r_1(o_1, a_1)\big)\Big)\\
&= \bigwedge_{a_3\in A_3} \Big(g_3(a_3) \to\downarrow_1\uparrow_1\downarrow_1 \big(\varphi_{1R^+}(\varphi_{2R}(a_3))\big)(o_1)\Big)\\
&= \bigwedge_{a_3\in A_3} \Big(g_3(a_3) \to\downarrow_1 \big((\varphi_2 \circ \varphi_1)_R(a_3)\big)(o_1)\Big)\\
&= \bigwedge_{a_3\in A_3} \Big(g_3(a_3) \to \bigwedge_{a_1\in A_1} \big((\varphi_2 \circ \varphi_1)_R(a_3)(a_1) \to r_1(o_1, a_1)\big)\Big)\\
&= \bigwedge_{a_1\in A_1} \bigwedge_{a_3\in A_3} \Big(\big(g_3(a_3) \otimes (\varphi_2 \circ \varphi_1)_R(a_3)(a_1)\big) \to r_1(o_1, a_1)\Big)\\
&= \bigwedge_{a_1\in A_1} \Big(\bigvee_{a_3\in A_3} (g_3(a_3) \otimes (\varphi_2 \circ \varphi_1)_R(a_3)(a_1)) \to r_1(o_1, a_1)\Big)\\
&=\downarrow_1 \big((\varphi_2 \circ \varphi_1)_{R^+}(g_3)\big)(o_1)
\end{aligned}
$$

$$
\begin{aligned}
\star =&\Big\langle \downarrow_1 ((\varphi_2 \circ \varphi_1)_{R^+}(g_3)), \uparrow_1\downarrow_1 ((\varphi_2 \circ \varphi_1)_{R^+}(g_3))\Big\rangle\\
=&(\varphi_2 \circ \varphi_1)_\wedge(\langle f_3, g_3\rangle)
\end{aligned}
$$

□

Proposition 1. *Γ is a functor from L-ChuCorrs to L-CLLOS.*

Proof. Straightforward from the previous lemmas. □

We continue by showing that the previous functor satisfies the conditions to define a categorical equivalence, characterized by the following result:

Theorem 3 (See [2]). *The following conditions on a functor $F : \mathcal{C} \to \mathcal{D}$ are equivalent:*

- *F is an equivalence of categories.*
- *F is full and faithful and "essentially surjective" on objects: for every $D \in \mathcal{D}$ there is some $C \in \mathcal{C}$ such that $F(C) \cong D$.*

Let us recall the definition of the notions required by the previous theorem:

Definition 14.

1. *A functor $F : \mathcal{C} \to \mathcal{D}$ is* faithful *if for all objects A, B of a category $\mathcal{C}$, the map $F_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ is injective.*
2. *Similarly, F is* full *if $F_{A,B}$ is always surjective.*

In our cases, for proving fullness and faithfulness of the functor $\varGamma$ we need to prove surjectivity and injectivity of the mapping

$$\varGamma_{C_1,C_2} : L\text{-ChuCors}(C_1, C_2) \to L\text{-CLLOS}(\varGamma(C_1), \varGamma(C_2))$$

for any two L-contexts C_1 and C_2. This will be done in the forthcoming lemmas.

Lemma 6. *$\varGamma$ is full.*

Proof. The point of the proof is to show that given any arrow $\langle s, z\rangle$ from the set L-CLLOS$(\varGamma(C_1), \varGamma(C_2))$ there exists an L-Chu correspondence $\varphi^{\langle s,z\rangle}$ from the set L-ChuCors(C_1, C_2), for any two L-contexts $C_i = \langle B_i, A_i, r_i\rangle$ for $i = \{1, 2\}$. Let us define the following mappings:

- $\varphi_L^{\langle s,z\rangle}(o_1) = Ext\big(s\big(\langle \downarrow_1\uparrow_1 (\chi_{o_1}), \uparrow_1 (\chi_{o_1})\rangle\big)\big)$
- $\varphi_R^{\langle s,z\rangle}(a_2) = Int\big(z\big(\langle \downarrow_2 (\chi_{a_2}), \uparrow_2\downarrow_2 (\chi_{a_2})\rangle\big)\big)$

$$\begin{aligned}
\uparrow_2 \big(\varphi_L^{\langle s,z\rangle}(o_1)\big)(a_2) &= \bigwedge_{o_2\in B_2} \big(\varphi_L^{\langle s,z\rangle}(o_1)(o_2) \to r_2(o_2, a_2)\big) \\
&= \bigwedge_{o_2\in B_2} \big(Ext(s(\langle \downarrow_1\uparrow_1 (\chi_{o_1}), \uparrow_1 (\chi_{o_1})\rangle))(o_2) \to \downarrow_2 (\chi_{a_2})(o_2)\big) \\
&= s\big(\langle \downarrow_1\uparrow_1 (\chi_{o_1}), \uparrow_1 (\chi_{o_1})\rangle\big) \preceq_2 \langle \downarrow_2 (\chi_{a_2}), \uparrow_2\downarrow_2 (\chi_{a_2})\rangle \\
&= \langle \downarrow_1\uparrow_1 (\chi_{o_1}), \uparrow_1 (\chi_{o_1})\rangle \preceq_1 z\big(\langle \downarrow_2 (\chi_{a_2}), \uparrow_2\downarrow_2 (\chi_{a_2})\rangle\big) \\
&= \bigwedge_{a_1\in A_1} \big(Int(z(\langle \downarrow_2 (\chi_{a_2}), \uparrow_2\downarrow_2 (\chi_{a_2})\rangle))(a_1) \to \uparrow_1 (\chi_{o_1})(a_1)\big) \\
&= \bigwedge_{a_1\in A_1} \big(\varphi_R^{\langle s,z\rangle}(a_2)(a_1) \to r_1(o_1, a_1)\big) \\
&= \downarrow_1 \big(\varphi_R^{\langle s,z\rangle}(a_2)\big)(o_1)
\end{aligned}$$

So $\varphi^{\langle s,z\rangle} \in L$-ChuCors$(C_1, C_2)$ and $\varGamma_{C_1,C_2}$ is surjective, hence $\varGamma$ is full. □

Lemma 7. *Γ is faithfull*

Proof. Now the point is to prove the injectivity of Γ_{C_1,C_2}.

Consider two L-Chu correspondences φ_1, φ_2 from L-ChuCors(C_1, C_2) such that $\varphi_1 \neq \varphi_2$, and let us fix the pair $(o_1, a_2) \in B_1 \times A_2$, such that

$$\uparrow_2 \big(\varphi_{1L}(o_1)\big)(a_2) = \downarrow_1 \big(\varphi_{1R}(a_2)\big)(o_1) \neq \uparrow_2 \big(\varphi_{2L}(o_1)\big)(a_2) = \downarrow_1 \big(\varphi_{2R}(a_2)\big)(o_1)$$

Let us assume that either $\downarrow_1 \big(\varphi_{1R}(a_2)\big)(o_1) > \uparrow_2 \big(\varphi_{2L}(o_1)\big)(a_2)$ or that both values from L are incomparable, that is equivalent to the following:

$$\downarrow_1 \big(\varphi_{1R}(a_2)\big)(o_1) \to \uparrow_2 \big(\varphi_{2L}(o_1)\big)(a_2) < 1$$

Now consider the L-concept $\langle \downarrow_1 (\varphi_{1R}(a_2)), \varphi_{1R}(a_2)\rangle$ and let us compare its images under the mappings $\varphi_{1\vee}$ and $\varphi_{2\vee}$.

$$\begin{aligned}
&\uparrow_2 \big(\varphi_{2L^+}\big(\downarrow_1 (\varphi_{1R}(a_2))\big)\big)(a_2)\\
&= \bigwedge_{o_2 \in B_2} \big(\varphi_{2L^+}(\downarrow_1 (\varphi_{1R}(a_2)))(o_2) \to r_2(o_2, a_2)\big)\\
&= \bigwedge_{o_2 \in B_2} \big(\bigvee_{b_1 \in B_1} \big(\varphi_{2L}(b_1)(o_2) \otimes \downarrow_1 (\varphi_{1R}(a_2))(b_1)\big) \to r_2(o_2, a_2)\big)\\
&= \bigwedge_{b_1 \in B_1} \big(\downarrow_1 (\varphi_{1R}(a_2))(b_1) \to \bigwedge_{o_2 \in B_2} \big(\varphi_{2L}(b_1)(o_2) \to r_2(o_2, a_2)\big)\big)\\
&= \bigwedge_{b_1 \in B_1} \big(\downarrow_1 (\varphi_{1R}(a_2))(b_1) \to \uparrow_2 (\varphi_{2L}(b_1))(a_2)\big)\\
&\leq \downarrow_1 (\varphi_{1R}(a_2))(o_1) \to \uparrow_2 (\varphi_{2L}(o_1))(a_2)\\
&< 1 \text{ because of the restriction given above}
\end{aligned}$$

Similarly, we can obtain:

$$\begin{aligned}
&\uparrow_2 (\varphi_{1L^+}(\downarrow_1 (\varphi_{1R}(a_2))))(a_2) =\\
&= \bigwedge_{b_1 \in B_1} \big(\downarrow_1 (\varphi_{1R}(a_2))(b_1) \to \uparrow_2 (\varphi_{1L}(b_1))(a_2)\big)\\
&= \bigwedge_{b_1 \in B_1} \big(\downarrow_1 (\varphi_{1R}(a_2))(b_1) \to \downarrow_1 (\varphi_{1R}(a_2))(b_1)\big) = 1
\end{aligned}$$

It means that $\varphi_{1\vee}\big(\langle\downarrow_1 (\varphi_{1R}(a_2)), \varphi_{1R}(a_2)\rangle\big) \neq \varphi_{2\vee}\big(\langle\downarrow_1 (\varphi_{1R}(a_2)), \varphi_{1R}(a_2)\rangle\big)$

Hence $\varphi_{1\vee}(\langle\downarrow_1 (\varphi_{1R}(a_2)), \varphi_{1R}(a_2)\rangle) \neq \varphi_{2\vee}(\langle\downarrow_1 (\varphi_{1R}(a_2)), \varphi_{1R}(a_2)\rangle)$ and $\varphi_{1\vee} \neq \varphi_{2\vee}$. So Γ_{C_1,C_2} is injective and Γ is faithfull. □

Proposition 2. *The functor Γ is an equivalence functor between L-ChuCors and L-CLLOS.*

Proof. Fullness and faithfulness of Γ is given by previous lemmas. Essential surjectivity on objects is ensured by the fact that given any object $\langle\langle V, \approx\rangle, \preceq\rangle$ of L-CLLOS there exists an L-context $\langle V, V, \preceq\rangle$, such that $\Gamma\big(\langle V, V, \preceq\rangle\big) \cong \langle\langle V, \approx\rangle, \preceq\rangle$. Hence, we can state that Γ is the functor of equivalence between L-ChuCors and L-CLLOS. □

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