Concept lattices in fuzzy relation equations*

Juan Carlos Díaz and Jesús Medina**

Department of Mathematics. University of Cádiz
Email: {juancarlos.diaz,jesus.medina}@uca.es

Abstract. Fuzzy relation equations are used to investigate theoretical and applicational aspects of fuzzy set theory, e.g., approximate reasoning, time series forecast, decision making and fuzzy control, etc.. This paper relates these equations to a particular kind of concept lattices.

1 Introduction

Recently, multi-adjoint property-oriented concept lattices have been introduced in [16] as a generalization of property-oriented concept lattices [10,11] to a fuzzy environment. These concept lattices are a new point of view of rough set theory [23] that considers two different sets: the set of objects and the set of attributes.

On the other hand, fuzzy relation equations, introduced by E. Sanchez [28], are associated to the composition of fuzzy relations and have been used to investigate theoretical and applicational aspects of fuzzy set theory [22], e.g., approximate reasoning, time series forecast, decision making, fuzzy control, as an appropriate tool for handling and modeling of nonprobabilistic form of uncertainty, etc. Many papers have investigated the capacity to solve (systems) of fuzzy relation equations, e.g., in [1, 8, 9, 25, 26].

In this paper, the multi-adjoint relation equations are presented as a generalization of the fuzzy relation equations [24,28]. This general environment inherits the properties of the multi-adjoint philosophy, consequently, e.g., several conjunctors and residuated implications defined on general carriers as lattice structures can be used, which provide more flexibility in order to relate the variables considered in the system.

Moreover, multi-adjoint property-oriented concept lattices and systems of multi-adjoint relation equations have been related in order to obtain results that ensure the existence of solutions in these systems. These definitions and results are illustrated by a toy example to improve the readability and comprehension of the paper.

Among all concept lattice frameworks, we have related the multi-adjoint property-oriented concept lattices to the systems of multi-adjoint relation equations, e.g., the extension and intension operators of this concept lattice can be

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** Corresponding author.

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used to represent multi-adjoint relation equations, and, as a result, the solutions of these systems of relation equations can be related to the concepts of the corresponding concept lattice.

The more important consequence is that this relation provides that the properties given, e.g., in [2–4,12,14,17,18,27] can be applied to obtain many properties of these systems. Indeed, it can be considered that the algorithms presented, e.g., in [5, 6, 15] obtain solutions for these systems.

The plan of this paper is the following: in Section 2 we will recall the multi-adjoint property-oriented concept lattices as well as the basic operators used and some properties; later, in Section 3, an example will be introduced to motivate the multi-adjoint relation equations. Once these equations have been presented, in Section 4 the multi-adjoint property-oriented concept lattices and the systems of multi-adjoint relation equations will be related in order to obtain results which ensure the existence of solutions in these systems; the paper ends with some conclusions and prospects for future work.

2 Multi-adjoint property-oriented concept lattices

The basic operators in this environment are the adjoint triples, which are formed by three mappings: a non-commutativity conjunctor and two residuated implications [13], which satisfy the well-known adjoint property.

Definition 1. *Let* $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ *be posets and* $\&\colon P_1 \times P_2 \to P_3$, $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$ *be mappings, then* $(\&, \swarrow, \nwarrow)$ *is an* adjoint triple *with respect to* P_1, P_2, P_3 *if:*

1. $\&$ *is order-preserving in both arguments.*
2. $\swarrow$ *and* $\nwarrow$ *are order-preserving on the first argument*[1] *and order-reversing on the second argument.*
3. $x \leq_1 z \swarrow y$ *iff* $x \,\&\, y \leq_3 z$ *iff* $y \leq_2 z \nwarrow x$, *where* $x \in P_1$, $y \in P_2$ *and* $z \in P_3$.

Example of adjoint triples are the Gödel, product and Łukasiewicz t-norms together with their residuated implications.

Example 1. Since the Gödel, product and Łukasiewicz t-norms are commutative, the residuated implications satisfy that $\swarrow^G = \nwarrow_G$, $\swarrow^P = \nwarrow_P$ and $\swarrow^L = \nwarrow_L$. Therefore, the Gödel, product and Łukasiewicz adjoint triples are defined on $[0, 1]$ as:

$$\&_P(x, y) = x \cdot y \qquad\qquad z \nwarrow_P x = \min(1, z/x)$$

$$\&_G(x, y) = \min(x, y) \qquad\qquad z \nwarrow_G x = \begin{cases} 1 & \text{if } x \leq z \\ z & otherwise \end{cases}$$

$$\&_L(x, y) = \max(0, x + y - 1) \qquad\qquad z \nwarrow_G x = \min(1, 1 - x + z)$$

[1] Note that the antecedent will be evaluated on the right side, while the consequent will be evaluated on the left side, as in logic programming framework.

In [19] more general examples of adjoint triples are given.

The basic structure, which allows the existence of several adjoint triples for a given triplet of lattices, is the multi-adjoint property-oriented frame.

Definition 2. *Given two complete lattices* $(L_1, \preceq_1)$ *and* $(L_2, \preceq_2)$*, a poset* $(P, \leq)$ *and adjoint triples with respect to* P, L_2, L_1*,* $(\&_i, \swarrow^i, \nwarrow_i)$*, for all* $i = 1, \ldots, l$*, a* multi-adjoint property-oriented frame *is the tuple*

$$(L_1, L_2, P, \preceq_1, \preceq_2, \leq, \&_1, \swarrow^1, \nwarrow_1, \ldots, \&_l, \swarrow^l, \nwarrow_l)$$

Multi-adjoint property-oriented frames are denoted as $(L_1, L_2, P, \&_1, \ldots, \&_l)$. Note that the notation is similar to a multi-adjoint frame [18], although the adjoint triples are defined on different carriers.

The definition of context is analogous to the one given in [18].

Definition 3. *Let* $(L_1, L_2, P, \&_1, \ldots, \&_l)$ *be a multi-adjoint property-oriented frame. A* context *is a tuple* (A, B, R, σ) *such that* A *and* B *are non-empty sets (usually interpreted as attributes and objects, respectively),* R *is a* P*-fuzzy relation* $R \colon A \times B \to P$ *and* $\sigma \colon B \to \{1, \ldots, l\}$ *is a mapping which associates any element in* B *with some particular adjoint triple in the frame.*[2]

From now on, we will fix a multi-adjoint property-oriented frame and context, $(L_1, L_2, P, \&_1, \ldots, \&_l)$, (A, B, R, σ).

Now we define the following mappings ${}^{\uparrow_\pi} \colon L_2^B \to L_1^A$ and ${}^{\downarrow^N} \colon L_1^A \to L_2^B$ as

$$g^{\uparrow_\pi}(a) = \sup\{R(a, b) \&_{\sigma(b)} g(b) \mid b \in B\} \tag{1}$$

$$f^{\downarrow^N}(b) = \inf\{f(a) \nwarrow_{\sigma(b)} R(a, b) \mid a \in A\} \tag{2}$$

Clearly, these definitions[3] generalize the classical possibility and necessity operators [11] and they form an isotone Galois connection [16]. There are two dual versions of the notion of Galois connetion. The most famous Galois connection, where the maps are order-reversing, is properly called *Galois connection*, and the other in which the maps are order-preserving, will be called *isotone Galois connection*. In order to make this contribution self-contained, we recall their formal definitions:

Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be posets, and ${}^{\downarrow} \colon P_1 \to P_2$, ${}^{\uparrow} \colon P_2 \to P_1$ mappings, the pair $({}^{\uparrow}, {}^{\downarrow})$ forms a *Galois connection* between P_1 and P_2 if and only if: ${}^{\uparrow}$ and ${}^{\downarrow}$ are order-reversing; $x \leq_1 x^{\downarrow\uparrow}$, for all $x \in P_1$, and $y \leq_2 y^{\uparrow\downarrow}$, for all $y \in P_2$.

The one we adopt here is the dual definition: Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be posets, and ${}^{\downarrow} \colon P_1 \to P_2$, ${}^{\uparrow} \colon P_2 \to P_1$ mappings, the pair $({}^{\uparrow}, {}^{\downarrow})$ forms an *isotone Galois connection* between P_1 and P_2 if and only if: ${}^{\uparrow}$ and ${}^{\downarrow}$ are order-preserving; $x \leq_1 x^{\downarrow\uparrow}$, for all $x \in P_1$, and $y^{\uparrow\downarrow} \leq_2 y$, for all $y \in P_2$.

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[2] A similar theory could be developed by considering a mapping $\tau \colon A \to \{1, \ldots, l\}$ which associates any element in A with some particular adjoint triple in the frame.

[3] From now on, to improve readability, we will write $\&_b$, $\nwarrow_b$ instead of $\&_{\sigma(b)}$, $\nwarrow_{\sigma(b)}$.

A concept, in this environment, is a pair of mappings $\langle g, f\rangle$, with $g \in L^B$, $f \in L^A$, such that $g^{\uparrow_\pi} = f$ and $f^{\downarrow^N} = g$, which will be called *multi-adjoint property-oriented formal concept*. In that case, g is called the *extension* and f, the *intension* of the concept. The set of all these concepts will be denoted as $\mathcal{M}_{\pi N}$ [16].

Definition 4. *A* multi-adjoint property-oriented concept lattice *is the set*

$$\mathcal{M}_{\pi N} = \{\langle g, f\rangle \mid g \in L_2^B, f \in L_1^A \text{ and } g^{\uparrow_\pi} = f, f^{\downarrow^N} = g\}$$

in which the ordering is defined by $\langle g_1, f_1\rangle \preceq \langle g_2, f_2\rangle$ *iff* $g_1 \preceq_2 g_2$ *(or equivalently* $f_1 \preceq_1 f_2$*).*

The pair $(\mathcal{M}_{\pi N}, \preceq)$ is a complete lattice [16], which generalize the concept lattice introduced in [7] to a fuzzy environment.

3 Multi-adjoint relation equations

This section begins with an example that motivates the definition of multi-adjoint relation equations, which will be introduced later.

3.1 Multi-adjoint logic programming

A short summary of the main features of multi-adjoint languages will be presented. The reader is referred to [20, 21] for a complete formulation.

A language $\mathcal{L}$ contains propositional variables, constants, and a set of logical connectives. In this fuzzy setting, the usual connectives are adjoint triples and a number of aggregators.

The language $\mathcal{L}$ is interpreted on a *(biresiduated) multi-adjoint lattice*,[4] $\langle L, \preceq, \swarrow^1, \nwarrow_1, \&_1, \dots, \swarrow^n, \nwarrow_n, \&_n\rangle$, which is a complete lattice L equipped with a collection of adjoint triples $\langle \&_i, \swarrow^i, \nwarrow_i\rangle$, where each $\&_i$ is a conjunctor intended to provide a *modus ponens*-rule with respect to $\swarrow^i$ and $\nwarrow_i$.

A *rule* is a formula $A \swarrow^i \mathcal{B}$ or $A \nwarrow_i \mathcal{B}$, where A is a propositional symbol (usually called the *head*) and $\mathcal{B}$ (which is called the *body*) is a formula built from propositional symbols $B_1, \dots, B_n$ ($n \geq 0$), truth values of L and conjunctions, disjunctions and aggregations. Rules with an empty body are called *facts*.

A *multi-adjoint logic program* is a set of pairs $\langle \mathcal{R}, \alpha\rangle$, where $\mathcal{R}$ is a rule and α is a value of L, which may express the confidence which the user of the system has in the truth of the rule $\mathcal{R}$. Note that the truth degrees in a given program are expected to be assigned by an expert.

Example 2. Let us to consider a multi-adjoint lattice

$$\langle [0, 1], \leq, \leftarrow_G, \&_G, \leftarrow_P, \&_P, \wedge_L\rangle$$

[4] Note that a multi-adjoint lattice is a particular case of a multi-adjoint property-oriented frame.

where $\&_G$ and $\&_P$ are the Gödel and product conjunctors, respectively, and $\leftarrow_G$, $\leftarrow_P$ their corresponding residuated implications. Moreover, the Łukasiewicz conjunctor $\wedge_L$ will be used in the program [13].

Given the set of variables (propositional symbols)

$$\Pi = \{\texttt{low\_oil}, \texttt{low\_water}, \texttt{rich\_mixture}, \texttt{overheating}, \texttt{noisy\_behaviour}, \texttt{high\_fuel\_consumption}\}$$

the following set of multi-adjoint rules form a multi-adjoint program, which may represent the behaviour of a motor.

$$\langle \texttt{high\_fuel\_consumption} \leftarrow_G \texttt{rich\_mixture} \wedge_L \texttt{low\_oil}, 0.8\rangle$$
$$\langle \texttt{overheating} \leftarrow_G \texttt{low\_oil}, 0.5\rangle$$
$$\langle \texttt{noisy\_behaviour} \leftarrow_P \texttt{rich\_mixture}, 0.8\rangle$$
$$\langle \texttt{overheating} \leftarrow_P \texttt{low\_water}, 0.9\rangle$$
$$\langle \texttt{noisy\_behaviour} \leftarrow_G \texttt{low\_oil}, 1\rangle$$

The usual procedural is to measure the levels of "oil", "water" and "mixture" of a specific motor, after that the values for `low_oil`, `low_water` and `rich_mixture` are obtained, which are represented in the program as facts, for instance, the next ones can be added to the program:

$$\langle \texttt{low\_oil} \leftarrow_P \top, 0.2\rangle$$
$$\langle \texttt{low\_water} \leftarrow_P \top, 0.2\rangle$$
$$\langle \texttt{rich\_mixture} \leftarrow_P \top, 0.5\rangle$$

Finally, the values for the rest of variables (propositional symbols) are computed [20]. For instance, in order to attain the value for $\texttt{overheating}(o, w)$, for a level of oil, o, and water, w, the rules $\langle \texttt{overheating} \leftarrow_G \texttt{low\_oil}, \vartheta_1\rangle$ and $\langle \texttt{overheating} \leftarrow_P \texttt{low\_water}, \vartheta_2\rangle$ are considered and its value is obtained as:

$$\texttt{overheating}(o, w) = (\texttt{low\_oil}(o) \,\&_G\, \vartheta_1) \vee (\texttt{low\_water}(w) \,\&_P\, \vartheta_2) \qquad (3)$$

Now, the problem could be to recompute the weights of the rules from experimental instances of the variables, that is, the values of `overheating`, `noisy_behaviour` and `high_fuel_consumption` are known for particular measures of `low_oil`, `low_water` and `rich_mixture`.

Specifically, given the levels of oil, $o_1, \dots, o_n$, the levels of water, $w_1, \dots, w_n$, and the measures of mixture, $t_1, \dots, t_n$, we may experimentally know the values of the variables: $\texttt{noisy\_behaviour}(t_i, o_i)$, $\texttt{high\_fuel\_consumption}(t_i, o_i)$ and $\texttt{overheating}(o_i, w_i)$, for all $i \in \{1, \dots, n\}$.

Considering Equation (3), the unknown elements could be ϑ_1 and ϑ_2 instead of $\texttt{overheating}(o, w)$. Therefore, the problem now is to look for the values of ϑ_1 and ϑ_2, which solve the following system obtained after assuming the experimental data for the propositional symbols, $ov_1, o_1, w_1, \dots, ov_n, o_n, w_n$.

$$\begin{array}{rcl}
\texttt{overheating}(ov_1) & = & (\texttt{low_oil}(o_1) \,\&_G\, \vartheta_1) \vee (\texttt{low_water}(w_1) \,\&_P\, \vartheta_2) \\
\vdots & \vdots & \qquad\vdots \qquad\qquad \vdots \\
\texttt{overheating}(ov_n) & = & (\texttt{low_oil}(o_n) \,\&_G\, \vartheta_1) \vee (\texttt{low_water}(w_n) \,\&_P\, \vartheta_2)
\end{array}$$

This system can be interpreted as a system of fuzzy relation equations in which several conjunctors, $\&_G$ and $\&_P$, are assumed. Moreover, these conjunctors could be neither non-commutative nor associative and defined in general lattices, as permit the multi-adjoint framework.

Next sections introduce when these systems have solutions and a novel method to obtain them using concept lattice theory.

3.2 Systems of multi-adjoint relation equations

The operators used in order to obtain the systems will be the generalization of the sup-$*$-composition, introduced in [29], and inf-$\rightarrow$-composition, introduced in [1]. From now on, a multi-adjoint property-oriented frame, $(L_1, L_2, P, \&_1, \dots, \&_l)$ will be fixed.

In the definition of a multi-adjoint relation equation an interesting mapping $\sigma\colon U \rightarrow \{1, \dots, l\}$ will be considered, which relates each element in U to an adjoint triple. This mapping will play a similar role as the one given in a multi-adjoint context, defined in the previous section, for instance, this map provides a partition of U in preference sets. A similar theory may be developed for V instead of U.

Let $U = \{u_1, \dots, u_m\}$ and $V = \{v_1, \dots, v_n\}$ be two universes, $R \in {L_2}^{U\times V}$ an unknown fuzzy relation, $\sigma\colon U \rightarrow \{1, \dots, l\}$ a map that relates each element in U to an adjoint triple, and $K_1, \dots, K_n \in P^U$, $D_1, \dots, D_n \in {L_1}^V$ arbitrarily chosen fuzzy subsets of the respective universes.

A *system of multi-adjoint relation equations with* sup-$\&$-*composition*, is the following system of equations

$$\bigvee_{u\in U} (K_i(u) \mathbin{\&_u} R(u, v)) = D_i(v), \quad i \in \{1, \dots, n\} \tag{4}$$

where $\&_u$ represents the adjoint conjunctor associated to u by σ, that is, if $\sigma(u) = (\&_s, \swarrow^s, \nwarrow_s)$, for $s \in \{1, \dots, l\}$, then $\&_u$ is exactly $\&_s$.

If an element v of V is fixed and the elements $K_i(u_j)$, $R(u_j, v)$ and $D_i(v)$ are written as k_{ij}, x_j and d_i, respectively, for each $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$, then System (4) can particularly be written as

$$\begin{array}{rcl} k_{11} \mathbin{\&_{u_1}} x_1 \vee \cdots \vee k_{1m} \mathbin{\&_{u_m}} x_m & = & d_1 \\ \vdots \qquad \vdots \qquad\qquad \vdots & & \vdots \\ k_{n1} \mathbin{\&_{u_1}} x_1 \vee \cdots \vee k_{nm} \mathbin{\&_{u_m}} x_m & = & d_n \end{array} \tag{5}$$

where k_{ij} and d_i are known and x_j must be obtained.

Hence, for each $v \in V$, if we solve System (5), then we obtain a "column" of R (i.e. the elements $R(u_j, v)$, with $j \in \{1, \dots, m\}$), thus, solving n similar systems, one for each $v \in V$, the unknown relation R is obtained.

Example 3. Assuming Example 2, in this case, we will try to solve the problem about to obtain the weights associated to the rules from particular observed data for the propositional symbols.

The propositional symbols (variables) will be written in short as: $\texttt{hfc}$, $\texttt{nb}$, $\texttt{oh}$, $\texttt{rm}$, $\texttt{lo}$ and $\texttt{lw}$, and the measures of particular cases of the behaviour of the motor will be: h_i, n_i, ov_i, r_i, o_i, w_i, for $\texttt{hfc}$, $\texttt{nb}$, $\texttt{oh}$, $\texttt{rm}$, $\texttt{lo}$ and $\texttt{lw}$, respectively, in each case i, with $i \in \{1, 2, 3\}$.

For instance, the next system associated to `overheating` is obtained from the computation provided in Example 2.

$$\begin{aligned}
\texttt{oh}(ov_1) &= (\texttt{lo}(o_1) \mathbin{\&_G} \vartheta^{\texttt{oh}}_{\texttt{lo}}) \vee (\texttt{lw}(w_1) \mathbin{\&_P} \vartheta^{\texttt{oh}}_{\texttt{lw}}) \\
\texttt{oh}(ov_2) &= (\texttt{lo}(o_2) \mathbin{\&_G} \vartheta^{\texttt{oh}}_{\texttt{lo}}) \vee (\texttt{lw}(w_2) \mathbin{\&_P} \vartheta^{\texttt{oh}}_{\texttt{lw}}) \\
\texttt{oh}(ov_3) &= (\texttt{lo}(o_3) \mathbin{\&_G} \vartheta^{\texttt{oh}}_{\texttt{lo}}) \vee (\texttt{lw}(w_3) \mathbin{\&_P} \vartheta^{\texttt{oh}}_{\texttt{lw}})
\end{aligned}$$

where $\vartheta^{\texttt{oh}}_{\texttt{lo}}$ and $\vartheta^{\texttt{oh}}_{\texttt{lw}}$ are the weights associated to the rules with head $\texttt{oh}$. Similar systems can be obtained to `high_fuel_consumption` and `noisy_behaviour`.

Assuming the multi-adjoint frame with carrier $L = [0, 1]$ and the Gödel and product triples, these systems are particular systems of multi-adjoint relational equations. The corresponding context is formed by the sets $U = \{\texttt{rm}, \texttt{lo}, \texttt{lw}, \texttt{rm} \wedge_L \texttt{lo}\}$, $V = \{\texttt{hfc}, \texttt{nb}, \texttt{oh}\}$; the mapping σ that relates the elements $\texttt{lo}$, $\texttt{rm} \wedge_L \texttt{lo}$ to the Gödel triple, and $\texttt{rm}$, $\texttt{lw}$ to the product triple; the mappings $K_1, \dots, K_n \in P^U$, defined as the values given by the propositional symbols in U on the experimental data, for instance, if $u = \texttt{lo}$, then $K_1(\texttt{lo}) = \texttt{lo}(o_1)$, …, $K_n(\texttt{lo}) = \texttt{lo}(o_n)$; and the mappings $D_1, \dots, D_n \in {L_1}^V$, defined analogously, for instance, if $v = \texttt{rm}$, then $D_1(\texttt{rm}) = \texttt{rm}(r_1)$, …, $D_n(\texttt{rm}) = \texttt{rm}(r_n)$.

Finally, the unknown fuzzy relation $R \in {L_2}^{U \times V}$ is formed by the weights of the rules in the program.

In the system above, $\texttt{oh}$ has been the element $v \in V$ fixed. Moreover, as there do not exist rules with body $\texttt{rm}$ and $\texttt{rm} \wedge_L \texttt{lo}$, that is, the weights for that hypothetical rules are 0, then the terms $(\texttt{rm}(r_i) \mathbin{\&_G} 0 = 0$ and $(\texttt{rm}(r_i) \wedge_L \texttt{lo}(o_i) \mathbin{\&_P} 0 = 0$ do not appear.

Its counterpart is a *system of multi-adjoint relation equations with* inf-$\nwarrow$-*composition*, that is,

$$\bigwedge_{v \in V} (R(u, v) \nwarrow_{u_j} K^*_j(v)) = E_j(u), \quad j \in \{1, \dots, m\} \tag{6}$$

considered with respect to unknown fuzzy relation $R \in {L_1}^{U \times V}$, and where $K^*_1, \dots, K^*_m \in P^V$ and $E_1, \dots, E_m \in {L_2}^U$. Note that $\nwarrow_{u_j}$ represents the corresponding adjoint implication associated to u_j by σ, that is, if $\sigma(u_j) = (\&_s, \swarrow^s, \nwarrow_s)$, for $s \in \{1, \dots, l\}$, then $\nwarrow_{u_j}$ is exactly $\nwarrow_s$. Remark that in System 6, the implication $\nwarrow_{u_j}$ does not depend of the element u, but of j. Hence, the implications used in each equation of the system are the same.

If an element u of U is fixed, fuzzy subsets $K^*_1, \dots, K^*_m \in P^V$, $E_1, \dots, E_m \in L_2^U$ are assumed, such that $K^*_j(v_i) = k_{ij}$, $R(u, v_i) = y_i$ and $E_j(u) = e_j$, for each $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$, then System (6) can particularly be written as

$$\begin{aligned}
y_1 \nwarrow_{u_1} k_{11} \wedge \cdots \wedge y_n \nwarrow_{u_1} k_{n1} &= e_1 \\
\vdots \qquad\quad \vdots \qquad\quad & \vdots \ \ \vdots \\
y_1 \nwarrow_{u_m} k_{1m} \wedge \cdots \wedge y_n \nwarrow_{u_m} k_{nm} &= e_m
\end{aligned} \tag{7}$$

Therefore, for each $u \in U$, we obtain a "row" of R (i.e. the elements $R(u, v_i)$, with $i \in \{1, \dots, n\}$), consequently, solving m similar systems, the unknown relation R is obtained.

Systems (5) and (7) have the same goal, searching for the unknown relation R although the mechanism is different.

Analyzing these systems, we have that the left side of these systems can be represented by the mappings $C_K\colon L_2^m \to L_1^n, I_{K^*}\colon L_1^n \to L_2^m$, defined as:

$$C_K(\bar{x})_i = k_{i1} \,\&_{u_1} x_1 \vee \cdots \vee k_{im} \,\&_{u_m} x_m, \text{ for all } i \in \{1, \dots, n\} \tag{8}$$

$$I_{K^*}(\bar{y})_j = y_1 \nwarrow_{u_j} k_{1j} \wedge \cdots \wedge y_n \nwarrow_{u_j} k_{nj}, \text{ for all } j \in \{1, \dots, m\} \tag{9}$$

where $\bar{x} = (x_1, \dots, x_m) \in L_2^m$, $\bar{y} = (y_1, \dots, y_n) \in L_1^n$, and $C_K(\bar{x})_i$, $I_{K^*}(\bar{y})_j$ are the components of $C_K(\bar{x})$, $I_{K^*}(\bar{y})$, respectively, for each $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$.

Hence, Systems (5) and (7) can be written as:

$$C_K(x_1, \dots, x_m) = (d_1, \dots, d_n) \tag{10}$$

$$I_{K^*}(y_1, \dots, y_n) = (e_1, \dots, e_m) \tag{11}$$

respectively.

4 Relation between multi-adjoint property-oriented concept lattices and multi-adjoint relation equation

This section shows that Systems (5) and (7) can be interpreted in a multi-adjoint property-oriented concept lattice. And so, the properties given to the isotone Galois connection ${}^{\uparrow_\pi}$ and ${}^{\downarrow^N}$, as well as to the complete lattice $\mathcal{M}_{\pi N}$ can be used in the resolution of these systems.

First of all, the environment must be fixed. Hence, a multi-adjoint context (A, B, S, σ) will be considered, such that $A = V'$, $B = U$, where V' has the same cardinality as V, σ will be the mapping given by the systems and $S\colon A \times B \to P$ is defined as $S(v_i', u_j) = k_{ij}$. Note that $A = V'$ is related to the mappings K_i, since $S(v_i', u_j) = k_{ij} = K_i(u_j)$;

Now, we will prove that the mappings defined at the end of the previous section are related to the isotone Galois connection. Given $\mu \in L_2^B$, such that $\mu(u_j) = x_j$, for all $j \in \{1, \dots, m\}$, the following equalities are obtained, for each $i \in \{1, \dots, n\}$:

$$\begin{aligned} C_K(\bar{x})_i &= k_{i1} \,\&_{u_1} x_1 \vee \cdots \vee k_{im} \,\&_{u_m} x_m \\ &= S(v_i', u_1) \,\&_{u_1} \mu(u_1) \vee \cdots \vee S(v_i', u_m) \,\&_{u_m} \mu(u_m) \\ &= \sup\{S(v_i', u_j) \,\&_{u_j} \mu(u_j) \mid j \in \{1, \dots, m\}\} \\ &= \mu^{\uparrow_\pi}(v_i') \end{aligned}$$

Therefore, the mapping $C_K\colon L_2^m \to L_1^n$ is equivalent to the mapping ${}^{\uparrow_\pi}\colon L_2^B \to L_1^A$, where an element $\bar{x}$ in L_2^m can be interpreted as a map μ in L_2^B, such that

$\mu(u_j) = x_j$, for all $j \in \{1, \dots, m\}$, and the element $C_K(\bar{x})$ as the mapping $\mu^{\uparrow_\pi}$, such that $\mu^{\uparrow_\pi}(v'_i) = C_K(\bar{x})_i$, for all $i \in \{1, \dots, n\}$.

An analogy can be developed applying the above procedure to mappings I_{K^*} and ${\downarrow}^N$, obtaining that the mappings $I_{K^*}\colon L_1^n \to L_2^m$ and ${\downarrow}^N\colon L_1^A \to L_2^B$ are equivalent.

As a consequence, the following result holds:

Theorem 1. *The mappings $C_K\colon L_2^m \to L_1^n, I_{K^*}\colon L_1^n \to L_2^m$, establish an isotone Galois connection. Therefore, $I_{K^*} \circ C_K\colon L_2^m \to L_2^m$ is a closure operator and $C_K \circ I_{K^*}\colon L_1^n \to L_1^n$ is an interior operator.*

As (C_A, I_{K^*}) is an isotone Galois connection, any result about the solvability of one system has its dual counterpart.

The following result explains when these systems can be solved and how a solution can be obtained.

Theorem 2. *System (5) can be solved if and only if $\langle \lambda_{\bar{d}}^{\downarrow^N}, \lambda_{\bar{d}} \rangle$ is a concept of $\mathcal{M}_{\pi N}$, where $\lambda_{\bar{d}}\colon A = \{v_1, \dots, v_n\} \to L_1$, defined as $\lambda_{\bar{d}}(v_i) = d_i$, for all $i \in \{1, \dots, n\}$. Moreover, if System (5) has a solution, then $\lambda_{\bar{d}}^{\downarrow^N}$ is the greatest solution of the system.*

Similarly, System (7) can be solved if and only if $\langle \mu_{\bar{e}}, \mu_{\bar{e}}^{\uparrow_\pi} \rangle$ is a concept of $\mathcal{M}_{\pi N}$, where $\mu_{\bar{e}}\colon B = \{u_1, \dots, u_m\} \to L_2$, defined as $\mu_{\bar{e}}(u_j) = e_j$, for all $j \in \{1, \dots, m\}$. Furthermore, if System (7) has a solution, then $\mu_{\bar{e}}^{\uparrow_\pi}$ is the smallest solution of the system.

The main contribution of the relation introduced in this paper is not only the above consequences, but a lot of other properties for Systems (5) and (7) that can be stabilized from the results proved, for example, in [2–4, 12, 14, 17, 18, 27].

Next example studies the system of multi-adjoint relation equations presented in Example 3.

Example 4. The aim will be to solve a small system in order to improve the understanding of the method. In the environment of Example 3, the following system will be solved assuming the experimental data: $\texttt{oh}(ov_1) = 0.5$, $\texttt{lo}(o_1) = 0.3$, $\texttt{lw}(w_1) = 0.3$, $\texttt{oh}(ov_2) = 0.7$, $\texttt{lo}(o_2) = 0.6$, $\texttt{lw}(w_2) = 0.8$, $\texttt{oh}(ov_3) = 0.4$, $\texttt{lo}(o_3) = 0.5$, $\texttt{lw}(w_3) = 0.2$.

$$\begin{aligned}
\texttt{oh}(ov_1) &= (\texttt{lo}(o_1) \mathbin{\&_G} \vartheta_{\texttt{lo}}^{\texttt{oh}}) \vee (\texttt{lw}(w_1) \mathbin{\&_P} \vartheta_{\texttt{lw}}^{\texttt{oh}})\\
\texttt{oh}(ov_2) &= (\texttt{lo}(o_2) \mathbin{\&_G} \vartheta_{\texttt{lo}}^{\texttt{oh}}) \vee (\texttt{lw}(w_2) \mathbin{\&_P} \vartheta_{\texttt{lw}}^{\texttt{oh}})\\
\texttt{oh}(ov_3) &= (\texttt{lo}(o_3) \mathbin{\&_G} \vartheta_{\texttt{lo}}^{\texttt{oh}}) \vee (\texttt{lw}(w_3) \mathbin{\&_P} \vartheta_{\texttt{lw}}^{\texttt{oh}})
\end{aligned}$$

where $\vartheta_{\texttt{lo}}^{\texttt{oh}}$ and $\vartheta_{\texttt{lw}}^{\texttt{oh}}$ are the variables.

The context is: $A = V' = \{1, 2, 3\}$, the set of observations, $B = U = \{\texttt{lo}, \texttt{lw}\}$, σ associates the propositional symbol `lo` to the Gödel triple and `lw` to the product triple. The relation $S\colon A \times B \to [0, 1]$ is defined in Table 1.

Therefore, considering the mapping $\lambda_{\texttt{oh}}\colon A \to [0, 1]$ associated to the values of `overheating` in each experimental case, that is $\lambda_{\texttt{oh}}(1) = 0.5$, $\lambda_{\texttt{oh}}(2) = 0.7$,

Table 1. Relation S.

	low_oil	low_water
1	0.3	0.3
2	0.6	0.8
3	0.5	0.2

and $\lambda_{\mathtt{oh}}(3) = 0.4$; and the mapping $C_K\colon [0,1]^2 \to [0,1]^3$, defined in Equation (8), the system above can be written as

$$C_K(\vartheta^{\mathtt{oh}}_{\mathtt{lo}}, \vartheta^{\mathtt{oh}}_{\mathtt{lw}}) = \lambda_{\mathtt{oh}}$$

Since, by the comment above, there exists $\mu \in [0,1]^B$, such that $C_K(\vartheta^{\mathtt{oh}}_{\mathtt{lo}}, \vartheta^{\mathtt{oh}}_{\mathtt{lw}}) = \mu^{\uparrow_\pi}$, the goal will be to attain the mapping $\mu \in [0,1]^B$, such that $\mu^{\uparrow_\pi} = \lambda_{\mathtt{oh}}$, which can be found if and only if $((\lambda_{\mathtt{oh}})^{\downarrow^N}, \lambda_{\mathtt{oh}})$ is a multi-adjoint property-oriented concept in the considered context, by Theorem 2.

First of all, we compute $(\lambda_{\mathtt{oh}})^{\downarrow^N}$.

$$\begin{aligned}
(\lambda_{\mathtt{oh}})^{\downarrow^N}(\mathtt{lo}) &= \inf\{\lambda_{\mathtt{oh}}(1) \nwarrow_G S(1,\mathtt{lo}), \lambda_{\mathtt{oh}}(2) \nwarrow_G S(2,\mathtt{lo}), \lambda_{\mathtt{oh}}(3) \nwarrow_G S(3,\mathtt{lo})\}\\
&= \inf\{0.5 \nwarrow_G 0.3, 0.7 \nwarrow_G 0.6, 0.4 \nwarrow_G 0.5\}\\
&= \inf\{1, 1, 0.4\} = 0.4\\
(\lambda_{\mathtt{oh}})^{\downarrow^N}(\mathtt{lw}) &= \inf\{0.5 \nwarrow_P 0.3, 0.7 \nwarrow_P 0.8, 0.4 \nwarrow_P 0.2\}\\
&= \inf\{1, 0.875, 1\} = 0.875
\end{aligned}$$

Now, the mapping $(\lambda_{\mathtt{oh}})^{\downarrow^N \uparrow_\pi}$ is obtained.

$$\begin{aligned}
(\lambda_{\mathtt{oh}})^{\downarrow^N \uparrow_\pi}(1) &= \sup\{S(1,\mathtt{lo}) \,\&_G (\lambda_{\mathtt{oh}})^{\downarrow^N}(\mathtt{lo}), S(1,\mathtt{lw}) \,\&_P (\lambda_{\mathtt{oh}})^{\downarrow^N}(\mathtt{lw})\}\\
&= \sup\{0.3 \,\&_G\, 0.4, 0.3 \,\&_P\, 0.875\}\\
&= \sup\{0.3, 0.2625\} = 0.3\\
(\lambda_{\mathtt{oh}})^{\downarrow^N \uparrow_\pi}(2) &= \sup\{0.6 \,\&_G\, 0.4, 0.8 \,\&_P\, 0.875\} = 0.7\\
(\lambda_{\mathtt{oh}})^{\downarrow^N \uparrow_\pi}(3) &= \sup\{0.5 \,\&_G\, 0.4, 0.2 \,\&_P\, 0.875\} = 0.4
\end{aligned}$$

Therefore, $((\lambda_{\mathtt{oh}})^{\downarrow^N}, \lambda_{\mathtt{oh}})$ is not a multi-adjoint property-oriented concept and thus, the considered system has no solution, although if the experimental value for `oh` had been 0.3 instead of 0.5, the system would have had a solution.

These changes could be considered in several applications where noisy variables exist and their values can be conveniently changed to obtain approximate solutions for the systems. Thus, if the experimental data for `overheating` are $\mathtt{oh}(ov_1) = 0.3$, $\mathtt{oh}(ov_2) = 0.7$ and $\mathtt{oh}(ov_2) = 0.4$, then the original system will have at least one solution and the values $\vartheta^{\mathtt{oh}}_{\mathtt{lo}}$, $\vartheta^{\mathtt{oh}}_{\mathtt{lw}}$ will be 0.4, 0.875, respectively for a solution. Consequently, the truth for the first rule is lower than for the second or it might be thought that it is more determinant in obtaining higher values for lw than for lo. Another possibility is to consider that this conclusion about the certainty of the rules is not correct, in which case another adjoint triple might be associate to lo.

As a result, the properties introduced in several fuzzy formal concept analysis frameworks can be applied in order to obtain solutions of fuzzy relation equations, as well as in the multi-adjoint general framework.

Furthermore, in order to obtain the solutions of Systems (5) and (7), the algorithms developed, e.g., in [5, 6, 15], can be used.

5 Conclusions and future work

Multi-adjoint relation equations have been presented that generalize the existing definitions presented at this time. In this general environment, different conjunctors and residuated implications can be used, which provide more flexibility in order to relate the variables considered in the system.

A toy example has been introduced in the paper in order to improve its readability and reduce the complexity of the definitions and results.

As a consequence of the results presented in this paper, several of the properties provided, e.g., in [2–4, 12, 14, 17, 18, 27], can be used to obtain additional characteristics of these systems.

In the future, we will apply the results provided in the fuzzy formal concept analysis environments to the general systems of fuzzy relational equations presented here.

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