# Links between modular decomposition of concept lattice and bimodular decomposition of a context

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**Abstract.** This paper is a preliminary attempt to study how modular and bimodular decomposition, used in graph theory, can be used on contexts and concept lattices in formal concept analysis (FCA).
In a graph, a module is a set of vertices defined in term of behaviour with respect to the outside of the module: All vertices in the module act with no distinction and can be replaced by a unique vertex, which is a representation of the module. This definition may be applied to concepts of lattices, with slighty modifications (using order relation instead of adjacency).
One can note that modular decomposition is not well suited for bipartite graphs. For example, every bipartite graph corresponding to a clarified context is trivially prime (not decomposable w.r.t modules). In [4], authors have introduced a decomposition dedicaced to bipartite graph, called the *bimodular decomposition*. In this paper, we show how modular decomposition of lattices and bimodular decomposition of contexts interact. These results may be used to improve readability of a Hasse diagram.

## 1 Introduction

Concept lattices are well suited to deal with knowledge representation and classification, but when the number of concepts grows, it is not very convenient to visualize the Hasse diagram. To avoid this problem, some approaches keep only part of the concepts (Iceberg lattice [9] , usage of Galois sub-hierarchy [3], concepts with high stability score [7, 8] or any combinaison of these techniques); Others approaches try to obtain a more readable lattice by usage of a threeshold, as for $\alpha$-galois lattices [10]. Another solution is to use decompositions to improve readability (see all chapter 4 of [5] and particularly nested diagrams).

There is a lot of works in graph theory about decomposition of a graph. A classical and well studied decomposition is the modular decomposition (see for example [6]). This decomposition has great properties: possibility of replacing a set of vertices by a single representant, so that visualization of the graph is better understandable; recursive approach, so that one can go from generalities to finer


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detail levels (useful for knowledge representation); nice theoretical properties, as the existence of a decomposition tree or closure properties for the family of modules.

Modular decomposition of graphs may be adapted to lattices with only little changes: In a graph, this is *the adjacency relation* which is fundamental, but it is *the order relation* for lattices.

Moreover, concept lattices are usually computed from a context, which can be considerated as a bipartite graph. So, there are two structures which can be decomposed: the concept lattice and the bipartite graph.

Unfortunately, bipartite graphs are not good candidates for modular decomposition: except for twin vertices (vertices with the same neighbourghood) or connected components, there are no modules (except trivial one's) in such graphs. To improve the decomposition of bipartite graph, the notion of *bimodule* is introduced in [4].

Goal of this paper is to study how bimodules of a bipartite graph interact with modules of the concept lattice of this context, and to see how it can be used to help the visualisation of information contained in lattices.

The next section is dedicaced to definitions. Section 2.2 introduces modules of a graph and transposes the definition to lattice (modules of a lattice). Section 3 is about bimodules of a bipartite graph (context) and the links that exist with the corresponding concept lattice. After some discussion in section 4, we conclude the paper in section 5.

## 2 Preliminaries

### 2.1 Definitions

In this paper, all discrete structures are finites and all graphs are simples (no loops neither multi-edges). Since this paper is about usages of graph theory results, a formal context will be considerated as a bipartite graph $B = (O, A, I)$ with $O$ (objects) and $A$ (attributes) being two stable sets of vertices, and $I$ (incidence relation between objects and attributes) the set of edges of $B$.

For a vertex $v$, $v'$ denotes the neighbourghood of $v$ (vertices adjacents to $v$). For a subset $V$ of vertices, $V'$ denotes the common neighbourhood (vertices which are adjacent to every vertices of $V$). With this notation, the classical definition of galois connections follows immediately.

**Definition 1 (Galois connections).** *For a set $X \subset O$, $Y \subseteq A$ we define*

$$X' = \{y \in O \mid xIy \text{ for all } x \in X\},$$
$$Y' = \{x \in A \mid xIy \text{ for all } y \in Y\}.$$

A clarified context is a context such that $x' = y'$ implie $x = y$ for any vertices of $O \cup A$. A clarified context is reduced if no vertex $v$ is such that $v' = V'$ with $V \subseteq O \cup A$, $v \notin V$.

A complete lattice $L = (P, \leq, \vee, \wedge)$ is a poset such that for all $X \subseteq P$, there exist a supremum and an infimum in $P$. $j \in P$ is $\vee$-irreducible element if $x \vee y = j$ implies $x = j$ or $y = j$. $m \in P$ is a $\wedge$-irreducible element if $x \wedge y = m$ implies $x = m$ or $y = m$. $j$ covers a unique element $j_*$ ($j_* \prec j$), $m$ is covered by a unique element $m^*$ ($m \prec m^*$). We denote $J$ the set of $\vee$-irreducible elements and $M$ the set of $\wedge$-irreducible elements.

For a formal context $C = (O, A, I)$ a formal concept is a pair $(X, Y)$, $X \subseteq O$, $X \subseteq A$ and $X' = Y$ and $Y' = X$. $X$ is called the extent of the concept and $Y$ is called the intent. The set of formal concepts ordered by inclusion on the intents is the concept lattice of $C$.

For every finite lattice $L = (P, \leq, \vee, \wedge)$ there is, up to isomorphism, a unique reduced context $C = (J, M, \leq)$. In the following of this paper, we will consider only reduced contexts, *i.e.* contexts such that $O = J$, the set of $\vee$-irreducible elements and $A = M$ the set of $\wedge$-irreducible elements of $L$.

### 2.2 Modules of graphs and lattices

We denote a graph $G$ with $G = (V, E)$. $V$ is the set of vertices and $E$ a set of edges. Let $X \subset V$ and $s \in V \backslash X$. Then $s$ *distinguishes* $X$ if $s' \cap X \neq \emptyset$ and $s' \cap X \neq X$. That is, $s$ is adjacent with some vertices of $X$ and not adjacent with some others vertices of $X$. So, if no vertex distinguishes a set $X$, then for the outside of $X$ and relation of adjacency, every vertex is similar and $X$ can be viewed as a unique vertex.

**Definition 2 (Module, graph theory).** *A module in a graph is a subset of vertices that no vertex distinguishes.*

The graph which is obtained by the replacement of a module by a single vertex is called a quotient graph. It is a simplification of the original one (see Fig. 1). As no vertex distinguishes $X$ (elements in dashed line), there exist only two possibilities for a vertex $v$ not in $X$: either $v$ is adjacent to every vertex of $X$ (then there exists an edge between $v$ and the representant of $X$) or $v$ is adjacent with no vertex of $X$ (then, there is no edge between $v$ and the representant of $X$).

For a graph $G = (V, E)$, the set $V$ and singletons $x \in V$ are trivial modules. A graph without non trivial module is called a prime graph (for the modular decomposition). Two modules $A$ and $B$ overlap if no one is a subset of the other and $A \cap B \neq \emptyset$. A module which does not overlap another module is a *strong module.*

Modules and strong modules are central in several decomposition processes and their properties have been well studied. In the first definitions, modules where defined with respect to the adjacency relation, but decompositions have been generalized (for example in [1]) for others properties of graphs.

For a lattice, it is more natural to consider the order relation than an adjacency relation, so a natural definition follows immediately:

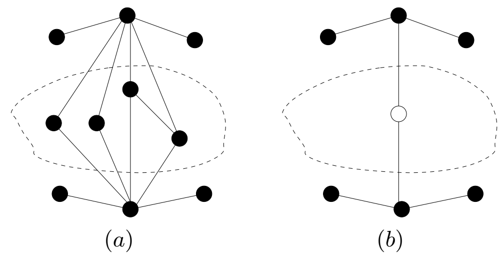



**Fig. 1.** $(a)$ A module in a graph and $(b)$ the quotient graph

**Definition 3.** *For a lattice* $L = (P, \leq, \vee, \wedge)$, *a lattice module is a set of elements* $X \subseteq P$ *such that, for every* $y \in P \backslash X$, *one of the three following statements is true:*

- $\forall x \in X,\ x < y$;
- *or* $\forall x \in X,\ x > y$;
- *or* $\forall x \in X,\ x || y$.

It is clear with this definition that a module in a lattice $L$ is equivalent to a module (with respect to adjacency) in the graph obtained by transitive closure of the Hasse Diagram of $L$.

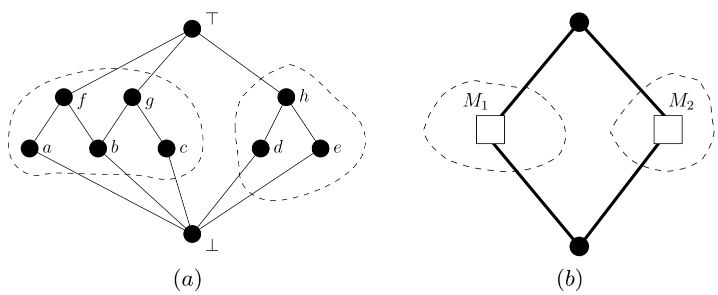



**Fig. 2.** Two strong modules of lattice $(a)$ and the quotient lattice $(b)$. Since no vertex outside the module distinguishes vertices inside the module, it can be collapsed to a single vertex which is the representant of the module. Note that $M_2$ can be recursively decomposed in two other modules $\{h\}$ (trivial) and $\{d, e\}$.

Let $X \subseteq P$ be a subset of elements of a lattice $L$, with $A = min(X)$ and $B = max(X)$ the sets of minimal (resp. maximal) elements of $X$. $X$ is a convex set iff for all $y \in P$ such that $a < y < b$, $a \in A$, $b \in B$, then $y \in X$. If $A$ and $B$ are reduced to singletons, $X$ is an interval. $[A, B]$ denotes the convex set defined by the two sets $A$ and $B$.

**Lemma 1.** *Modules in lattices are convex sets.*

*Proof.* Suppose it is not, then there exists $y \in P \backslash X$ with $a < y < b$ and so, $y$ distinguishes $a$ and $b$. It follows that $X$ is not a module.

From now, since lattices modules are convex sets, we will use the notation $X = [A, B]$ to speak of a module $X$.

**Lemma 2.** *For a lattice module* $[A, B]$*:*

1. *if* $|A| > 1$ *then* $A \subseteq J$*,*
2. *if* $|B| > 1$ *then* $B \subseteq M$*.*

*Proof.* Suppose $|A| > 1$, and let $A = a_1, a_2, \ldots, a_n$.
Suppose $a_i \notin J$, then, since $a_i || a_j$ there exists at least one $\vee$-irreducible element $j$ such that $j < a_i$ and $j \nless a_j$, which is a contradiction with the fact that $[A, B]$ is a module.

Dually proof applies for elements of $B$.

Note that when $|A| = 1$ (dually for $B$) the maximal element of the module is not necessary an irreducible one (See Fig. 3).

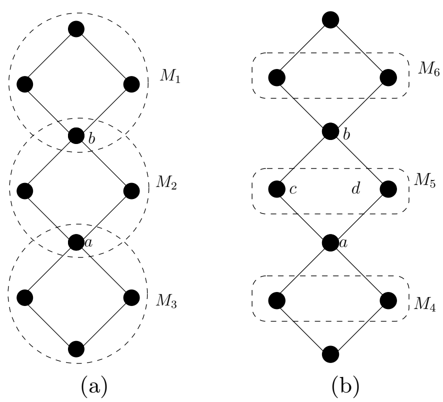



**Fig. 3.** (*a*) Module $M_2$ is a convex set $[A, B]$, with $A = \{a\} \not\subset J$ and $B = \{b\} \not\subset M$. $M_1$, $M_2$ and $M_3$ overlap: there are not strong modules. (*b*) $M_4$, $M_5$ and $M_6$ are strong modules but are not intervals. $M_5 = [A, B]$, with $A = \{b, c\} \subset J$ and $B = \{b, c\} \subset M$.

**Lemma 3.** *For a lattice module* $[A, B]$*:*

1. *if* $|A| > 2$, $\bigwedge A = a_i \wedge a_j$ *for all* $a_i, a_j \in A$*.*
2. *if* $|B| > 2$, $\bigvee B = b_i \vee b_j$ *for all* $b_i, b_j \in B$*.*

*Proof.* Clearly, suppose $|A| > 2$ and there exist $a_i, a_j, a_k \in A$ such that $x_1 = a_i \wedge a_j \neq a_j \wedge a_k = x_2$. w.l.o.g suppose $x_1 \nless x_2$. Then $x_1 < a_j$ and $x_1 \nless a_k$. It follows that $x_1$ distinguishes $[A, B]$.

## 3 Modules of lattices and bimodules of contexts

As a preliminary remark, we recall that all considered contexts are reduced, and so, clarified. The clarification of a context is the fact to keep only one object $o$ for all objects $o_i$ such that $o'_i = o'_j$ (dually for attribute). It is clear that the set $\{o_1, \ldots, o_n\}$ is a module in the bipartite graph and this process is equivalent to replace twin vertices by a representant.

Modules are not well situed for bipartite graphs. Twin vertices and connected components are the only modules for these graphs which are poorly decomposable. In goal to improve the decomposition, Fouquet and *all* have introduced *bimodule*, an analog of module for bipartite graphs.

**Definition 4 (Bimodule).** *Let $C = (O, A, I)$ be a bipartite graph, and $(X, Y) \subset (O, A)$, then $(X, Y)$ is a bimodule if no $x \in O \backslash X$ distinguishes $A$ and no $y \in A \backslash Y$ distinguishes $O$.*

Example of bimodule is given in Fig. 4: $b$ and $c$ are not distinguished with respect to vertices 4 (none of them are adjacent) or 3 (each of them is adjacent). Similarly, 1 and 2 are not distinguished by $a$ (each of them is adjacent) and $d$ (none of them is adjacent).

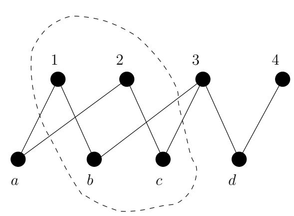



**Fig. 4.** Example of bimodules

The whole bipartite $(O, A)$, all vertices and pairs $(j, m)$, $j \in J$, $m \in M$ are trivial modules. In the following, we consider only non trivial bimodules, *i.e* bimodules with at least 3 elements.

**Proposition 1.** *To any non trivial module $X$ of lattice corresponds a bimodule of reduced context.*

*Proof.* By definition of a lattice module, no elements inside the modules are distinguished by elements outside. It follows directly that no $\vee$-irreducible element outside the module distinguishes $\wedge$-irreducible elements inside, and conversely.

Now, we want to know how a bimodule on the context may be interpretated in the concept lattice. First, we define a set in the lattice $L$ from a bimodule $X$.

From a bimodule $X = (J_1, M_1) \subseteq (J, M)$, we build a subset $C$ of concepts in $L$ such that:

– attributes concepts of $X$ are in $C$,
– objects concepts of $X$ are in $C$,
– $C = [A, B]$ is a convex set, with $A$ being maximal elements of $C$ and $B$ being minimal elements of $C$.

As previously seen, a lattice module corresponds to a context bimodule but, with the previous construction, the converse may be false: There exist bimodules of a context such that $[A, B]$ does not correspond to a module in the lattice. As an example, Fig. 5 shows the lattice for the bipartite graph in Fig. 4. The set $[A, B]$ is bounded by a dashed line.

Nevertheless, we can observe that, even if $[A, B]$ is not a module, there exists a possibility of simplification, replacing a set of elements by two vertices and an edge.

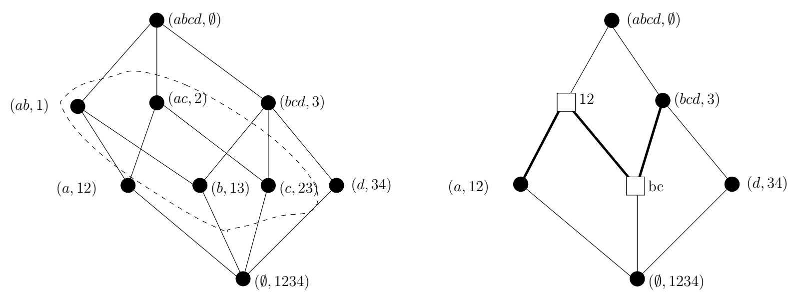



**Fig. 5.** ($a$) lattice for bipartite graph in Fig. 4.$a$ and ($b$) simplified lattice

In the following, we show that when $[A, B]$ is not a module, the shape of the set $[A, B]$ is very constrained.

**Proposition 2.** *Let $[A, B]$ be a convex set built from a non trivial bimodule $X$. If $[A, B]$ is not a module, then*

1. $|\ |[A, B] \cap M| - |[A, B] \cap J|\ | \leq 1$
2. *if* $|[A, B] \cap M| = |[A, B] \cap J|$, *then* $A \subseteq J$ *and* $B \subseteq M$
3. *if* $|max([A, B] \cap J)| > 1$, $a_+ = \bigvee a_i$, $a_i \in max([A, B] \cap J)$ *is such that* $a_i \prec a_+$
4. *if* $|min([A, B] \cap M)| > 1$, $b_- = \bigwedge b_i$, $b_i \in min([A, B] \cap M)$ *is such that* $b_- \prec b_i$

*Proof.* First, we show that $|\ |[A, B] \cap M| - |[A, B] \cap J|\ | \leq 1$:

Suppose that $[A, B]$ is not a module of the concept lattice, then there exists an element $x \notin [A, B]$ which distinguishes $[A, B]$. Without lost of generality, we can consider that $x \in J$ and there exist $y \in [A, B]$ such that $x < y$. We denote $P_1$

the set of elements of $[A,B]$ which are greater than $x$ and $P_2 = [A,B]\backslash P_1$. Since $x$ is a $\vee$-irreducible element, it does not distinguish any $\wedge$-irreducible elements in $[A,B]$. It follows that all $\wedge$-irreducible elements in $[A,B]$ are in $P_1$ and none of them in $P_2$ (or conversely).

In a finite lattice, every element $e$ is $\wedge$-dense, *i.e.* equal to the infimum of $\wedge$-irreducible elements greater than $e$.

All $\vee$-irreducible elements in $P_2$ cannot be distinguished by $\wedge$-irreducible elements outside $[A,B]$. One unique $\vee$-irreducible element $j_{max}$ of $P_2$ may be defined by $j_{max} = \bigwedge m_i, \ldots, m_j$, with $m_i, \ldots, m_j \notin P_1$. All other $\vee$-irreducible elements in $P_2$ are distinguished by $\wedge$-irreducible elements in $P_1$ (and only by these elements). So $P_2 \cap J = X \cup \{j_{max}\}$ ($j_{max}$ may not exist).

Suppose $|X| < |[A,B] \cap M|$, then there exist $m_1, m_2 \in [A,B] \cap M$, $j_1 \in X$ such that $j_1 < m_1$, $j_1 < m_2$, $m_1 || m_2$. It follows that $j_1 < m_1 \wedge m_2$, which is impossible since $j_1 \in P_2$ and elements in $P_2$ are not comparable to $x$.

Similarly, suppose $|X| > |[A,B] \cap M|$, At least one $\vee$-irreducible element $j$ of $X$ is smaller than two $\wedge$-irreducible elements $m_1$ and $m_2$ of $P_1$, with $m_1 || m_2$. This is impossible, so $|X| = |[A,B] \cap M|$ and $|\ |[A,B] \cap M| - |[A,B] \cap J|\ | \leq 1$.

$A \subseteq J$ and $B \subseteq M$ follow directly of the fact that, by construction $A$ and $B$ contain irreducible elements and for each $\wedge$-irreducible element $m \in [A,B]$, there exists a $\vee$-irreducible element $j \in [A,B]$ such that $j < m$.

It remains to prove that, when $max([A,B] \cap J)$ contains at least two elements, $a_+ = \bigvee a_i$, $a_i \in max([A,B] \cap J)$ is such that $a_i \prec a_+$ (and dually for $b_-$). Suppose it is not the case, then exist at least two elements $x_1$ and $x_2$ smaller than $a_+$ and such that $x_1$ and $x_2$ distinguish elements in $A$. It follows that one can find a $\wedge$-irreducible element which distinguishes $\vee$-irreducible elements in $A$ and that is a contradiction.

It follows from this proposition that even if a set $[A,B]$ is not a module, it can be collapsed into two vertices $j$ and $m$ such that $j < m$ (but maybe not $j \prec m$). $j$ is a representant for the set $[A,B] \cap J$ and $m$ a representant for the set $[A,B] \cap M$. Moreover, $j \prec a_+$ and $b_+ \prec m$.

## 4 Discussion

### 4.1 Algorithmic Aspects

It is known that the family of modules of a graph (and so, of a lattice) and the family of bimodules of a bipartite graph are closed by intersection. Since the whole graph is a (trivial) module, it defines a lattice. So, for any set $S$ of vertices, it is possible to use a closure operator to compute the smallest module which contains $S$. Algorithm 1 adds all vertices which distinguish respectively $X$ and $Y$ and the same process is repeated until no more vertex can be added.

Usually, bimodules decomposition does not produce all possible modules, but an inclusion tree such that all possible bimodules can be deduced from this tree. The root represents the whole graph and the leaves are vertices (trivial

**Input**: $(O, A, I)$ a bipartite graph, $(X, Y) \subset (O, A)$
**Output**: $(X_c, Y_c)$, smallest bimodule containing $(X, Y)$

```
begin
    continue ← true;
    (X_c, Y_c) ← (X, Y);
    while continue do
        continue ← false;
        forall the x ∈ J\X_c do
            if x distinguishes Y_c then
                X_c ← X_c ∪ x;
                continue ← true;
            end
        end
        forall the y ∈ M\Y_c do
            if y distinguishes X_c then
                Y_c ← Y_c ∪ y;
                continue ← true;
            end
        end
    end
    return (X_c, Y_c)
end
```

**Algorithm 1:** Computation of the smallest bimodule which contains $(X, Y)$

bimodules). It follows that the size of the tree is $O(n)$, with $n = |O| + |A|$. In [1], authors propose a $O(n^3)$ algorithm to compute a such tree.

### 4.2 Decomposition and Real Data

In Fig. 6, an example of bimodule is shown on the "Living Beings and Water" concept lattice [5]. $g$ is the attribute for "can move around" and $h$ is the one for "has limbs". These two attributes are equivalent (cannot be distinguished) from the outside of the bimodule. So, on the lattice in Fig. 6.c these two attributes are collapsed, as well as objects 1 (Leech) and 2 (Bream). Further work must be done on real data to see what bimodules can enlight for practical cases.

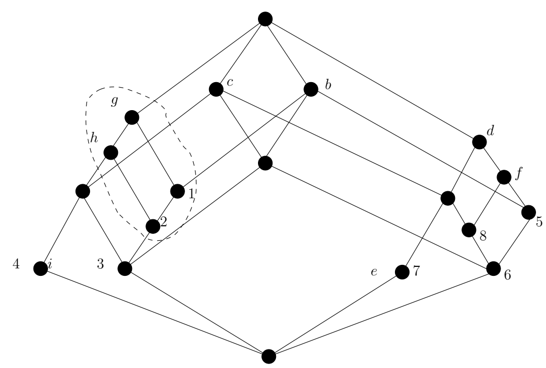
## 5 Conclusion

First, we have seen that modules defined on a lattice have natural links with bimodules of the bipartite graph (context) of this lattice. Modules of a lattice can be used the same way as modules of a graph are used: to produce a quotient lattice, which is a simplification of the original one. Recursive definition of modules allows to consider several details levels in the lattice.

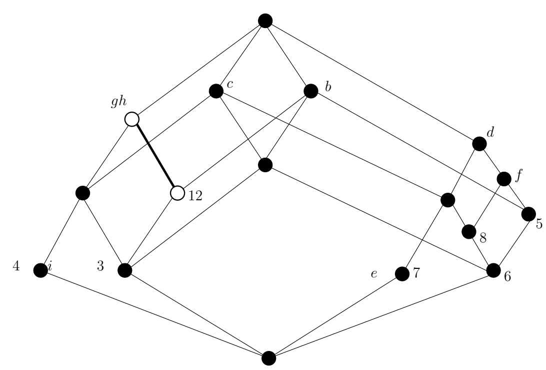
All results in modular decomposition may be transposed immediatly to concept lattice and associated context to improve the readibility of the lattice.

| | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ | $i$ |
|---|---|---|---|---|---|---|---|---|
| 1 | × | | | | | × | | |
| 2 | × | | | | | × | × | |
| 3 | × | × | | | | × | × | |
| 4 | | × | | | | × | × | × |
| 5 | × | | × | | × | | | |
| 6 | × | × | × | | × | | | |
| 7 | | × | × | × | | | | |
| 8 | | × | × | | × | | | |

($a$)




($b$)




($c$)

**Fig. 6.** ($a$) "Living Beings and Water " Context [5], ($b$) Concept lattice for "living Beings and Water" and ($c$) the same concept lattice with a bimodule collapsed.

Second, investigation of bimodules properties shows that a bimodule may not correspond to a module of the lattice. Nevertheless, it remains possible to use it to produce a simplification of the original lattice. In such a case, the bimodule is collapsed in two elements $a$ and $b$ which represent $\vee$-irreducible elements and $\wedge$-irreducible elements of the bimodule.

This last case is a particular case of another decomposition proposed for inheritance hierarchies [2], called the block decomposition (with a different definition of block that the one in [5]): a block is an interval $[a, b]$ such that only $a$ and $b$ can be distinguished of other vertices from the outside of the block. As a perspective behaviour of this decomposition for lattices and associated properties on the context can be investigated.

## References


1. m. Bui Xuan, B., Habib, M., Limouzy, V., Montgolfier, F.D.: Homogeneity vs. adjacency: generalising some graph decomposition algorithms. In: In 32nd International Workshop on Graph-Theoretic Concepts in Computer Science (WG), volume 4271 of LNCS (2006)
2. Capelle, C.: Block decomposition of inheritance hierarchies. In: WG. pp. 118–131 (1997)
3. Encheva, S.: Galois sub-hierarchy and orderings. In: Proceedings of the 10th WSEAS international conference on Artificial intelligence, knowledge engineering and data bases. pp. 168–171. AIKED'11, World Scientific and Engineering Academy and Society (WSEAS), Stevens Point, Wisconsin, USA (2011), http://portal.acm.org/citation.cfm?id=1959485.1959517
4. Fouquet, J., Habib, M., de Montgolfier, F., Vanherpe, J.: Bimodular decomposition of bipartite graphs. In: Graph-Theoretic Concepts in Computer Science 30th International Workshop, WG 2004, Bad Honnef, Germany, June 21-23 (2004)
5. Ganter, B., Wille, R.: Formal Concept Analysis, Mathematical Foundations. Springer-Verlag Berlin (1996)
6. Habib, M., Paul, C.: A survey of the algorithmic aspects of modular decomposition. Computer Science Review 4, 41–59 (2010)
7. Jay, N., Kohler, F., Napoli, A.: Analysis of social communities with iceberg and stability-based concept lattices. In: Proceedings of the 6th international conference on Formal concept analysis. pp. 258–272. ICFCA'08, Springer-Verlag, Berlin, Heidelberg (2008), http://portal.acm.org/citation.cfm?id=1787746.1787765
8. Klimushkin, M., Obiedkov, S., Roth, C.: Approaches to the selection of relevant concepts in the case of noisy data. In: Kwuida, L., Sertkaya, B. (eds.) Proc. 8th Intl. Conf. Formal Concept Analysis. LNCS/LNAI, vol. 5986, pp. 255–266. Springer (2010)
9. Stumme, G., Taouil, R., Bastide, Y., Lakhal, L.: Conceptual clustering with iceberg concept lattices. In: In: Proc. of GI-Fachgruppentreffen Maschinelles Lernen'01, Universität Dortmund (2001)
10. Ventos, V., Soldano, H.: Alpha galois lattices: an overview. In: In: International Conference in Formal Concept Analysis (ICFCA05), LNCS. pp. 298–313. Springer (2005)