Fuzzy-Valued Triadic Implications

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Abstract. We present a new approach for handling fuzzy triadic data in the setting of Formal Concept Analysis. The starting point is a fuzzy-valued triadic context (K_1, K_2, K_3, Y), where K_1, K_2 and K_3 are sets and Y is a ternary fuzzy relation between these sets. First, we generalise the methods of Triadic Concept Analysis to our setting and show how they fit other approaches to Fuzzy Triadic Concept Analysis. Afterwards, we develop the fuzzy-valued triadic implications as counterparts of the various triadic implications studied in the literature. These are of major importance for the integrity of Fuzzy and Fuzzy-Valued Triadic Concept Analysis.

Keywords: Formal Concept Analysis, fuzzy data, three-way data

1 Introduction

So far, the fuzzy approaches to Triadic Concept Analysis considered all three components of a triadic concept as fuzzy sets. In [1] the methods from Triadic Concept Analysis were generalised to the fuzzy setting. A more general approach was presented in [2], where different residuated lattices were considered for each fuzzy set. A somehow different strategy was considered in [3] using alpha-cuts.

Our approach differs from the other ones in considering just two components as fuzzy and one as crisp in a triadic concept. This is motivated by the fact that in some situations it is not appropriate to regard all sets as fuzzy. For example, it is not natural to say that *half of a person is old*, however we may say *a person is half old*. First, we translate methods of Triadic Concept Analysis to our setting. Compared to other works, we generalise all triadic derivation operators and show how they change for the fuzzy approaches considered by other authors. Besides these results, the main achievement of this paper is the generalisation of the various triadic implications presented in [4].

Due to the large amount of results in this paper, we concentrate on giving an intuition of the methods and omit proofs whenever they do not influence the understanding. The missing proofs, further results concerning fuzzy-valued triadic concepts and trilattices can be found in [5]. There, we also study the fuzzy-valued triadic approach to Factor Analysis.

The paper is structured as follows: In Section 2 we give brief introductions to Triadic and Formal Fuzzy Concept Analysis. In Section 3 we develop our

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fuzzy-valued setting, defining context, concept, derivation operators and show how they correspond other to approaches to Fuzzy Triadic Concept Analysis. We also comment on the reasons why our setting is a proper generalisation. In Section 4 we present the fuzzy-valued triadic implications. The developed methods are accompanied by illustrative examples. The last section contains concluding remarks and further topics of research.

2 Preliminaries

We assume basic familiarities with Formal Concept Analysis and refer the reader to [6]. In the following we give brief introductions to Triadic Concept Analysis [7,8] and Formal Fuzzy Concept Analysis [9,10].

2.1 Triadic Concept Analysis

As introduced in [7], the underlying structure of Triadic Concept Analysis is a **triadic context** defined as a quadruple (K_1, K_2, K_3, Y) where K_1, K_2 and K_3 are sets and Y is a ternary relation, i.e., $Y \subseteq K_1 \times K_2 \times K_3$. The elements of K_1, K_2 and K_3 are called **(formal) objects, attributes** and **conditions**, respectively, and $(g, m, b) \in Y$ is read: *object g has attribute m under condition b*. A **triadic concept** (shortly **triconcept**) of a triadic context (K_1, K_2, K_3, Y) is defined as a triple (A_1, A_2, A_3) with $A_i \subseteq K_i$, $i \in \{1, 2, 3\}$ that is maximal with respect to component-wise set inclusion. For a triconcept (A_1, A_2, A_3), the components A_1, A_2 and A_3 are called the **extent**, the **intent**, and the **modus** of (A_1, A_2, A_3), respectively.

Small triadic contexts can be represented through three-dimensional cross tables (see Example 1). Pictorially, a triconcept is a rectangular box full of crosses in the three-dimensional cross table representation of (K_1, K_2, K_3, Y), where this "box" is maximal under proper permutation of rows, columns and layers of the cross table.

For $\{i, j, k\} = \{1, 2, 3\}$ with $j < k$ and for $X \subseteq K_i$ and $Z \subseteq K_j \times K_k$, the $(-)^{(i)}$**-derivation operators** are defined by

$$X \mapsto X^{(i)} := \{(k_j, k_k) \in K_j \times K_k \mid (k_i, k_j, k_k) \in Y \text{ for all } k_i \in X\}, \quad (1)$$

$$Z \mapsto Z^{(i)} := \{k_i \in K_i \mid (k_i, k_j, k_k) \in Y \text{ for all } (k_j, k_k) \in Z\}. \quad (2)$$

These derivation operators correspond to the derivation operators of the dyadic contexts defined by $\mathbb{K}^{(i)} := (K_i, K_j \times K_k, Y^{(i)})$ for $\{i, j, k\} = \{1, 2, 3\}$, where $k_1 Y^{(1)} (k_2, k_3) :\Longleftrightarrow k_2 Y^{(2)} (k_1, k_3) :\Longleftrightarrow k_3 Y^{(3)} (k_1, k_2) :\Longleftrightarrow (k_1, k_2, k_3) \in Y$. Due to the structure of triadic contexts further derivation operators can be defined. For $\{i, j, k\} = \{1, 2, 3\}$ and for $X_i \subseteq K_i$, $X_j \subseteq K_j$ and $X_k \subseteq K_k$ the $(-)^{X_k}$**-derivation operators** are defined by

$$X_i \mapsto X_i^{X_k} := \{k_j \in K_j \mid (k_i, k_j, k_k) \in Y \text{ for all } (k_i, k_k) \in X_i \times X_k\}, \quad (3)$$

$$X_j \mapsto X_j^{X_k} := \{k_i \in K_i \mid (k_i, k_j, k_k) \in Y \text{ for all } (k_j, k_k) \in X_j \times X_k\}. \quad (4)$$

These derivation operators correspond to the derivation operators of the dyadic contexts defined by $\mathbb{K}^{ij}_{X_k} := (K_i, K_j, Y^{ij}_{X_k})$ where $(k_i, k_j) \in Y^{ij}_{X_k}$ if and only if $(k_i, k_j, k_k) \in Y$ for all $k_k \in X_k$. The structure on the set of all triconcepts $\mathfrak{T}(\mathbb{K})$ is the set inclusion in each component of the triconcept. For each $i \in \{1, 2, 3\}$ there is a quasiorder $\lesssim_i$ and its corresponding equivalence relation $\sim_i$ defined by

$$(A_1, A_2, A_3) \lesssim_i (B_1, B_2, B_3) :\Longleftrightarrow A_i \subseteq B_i \text{ and}$$
$$(A_1, A_2, A_3) \sim_i (B_1, B_2, B_3) :\Longleftrightarrow A_i = B_i \ (i = 1, 2, 3).$$

The triconcepts ordered in this way form complete *trilattices*, the triadic counterparts of concept lattices, as proved in the Basic Theorem of Triadic Concept Analysis [8]. However, unlike the dyadic case, the extents, intents and modi, respectively, do not form a closure system in general.

Example 1. The triadic context displayed below consists of the object set $K_1 = \{1, 2, 3\}$, the attribute set $K_2 = \{a, b, c\}$ and the condition set $K_3 = \{A, B\}$. The context has 12 triconcepts which are displayed in the same figure on the right. For example, the first concept means that object 1 has attributes a and b under

	A			B		
	a	b	c	a	b	c
1	×	×		×	×	
2		×	×	×		×
3		×	×	×	×	×

No.	Extent	Intent	Modus	No.	Extent	Intent	Modus
1	$\{1\}$	$\{a, b\}$	$\{K_3\}$	7	$\{3\}$	$\{K_2\}$	$\{B\}$
2	$\{K_1\}$	$\{b\}$	$\{A\}$	8	$\{K_1\}$	$\{a\}$	$\{B\}$
3	$\{2, 3\}$	$\{b, c\}$	$\{A\}$	9	$\{2, 3\}$	$\{c\}$	$\{K_3\}$
4	$\{\varnothing\}$	$\{K_2\}$	$\{K_3\}$	10	$\{3\}$	$\{b, c\}$	$\{K_3\}$
5	$\{1, 3\}$	$\{a, b\}$	$\{B\}$	11	$\{K_1\}$	$\{K_2\}$	$\{\varnothing\}$
6	$\{2, 3\}$	$\{a, c\}$	$\{B\}$	12	$\{K_1\}$	$\{\varnothing\}$	$\{K_3\}$

Fig. 1. Triadic context and the associated triconcepts

all conditions from K_3. However, as two components of a triconcept are necessary to determine the third one, $\{a, b\}$ is also an intent of another triconcept, namely of the fifth one.

2.2 Formal Fuzzy Concept Analysis

A **complete residuated lattice** $\mathbf{L} := (L, \wedge, \vee, \otimes, \to, 0, 1)$ is an algebra such that: (1) $(L, \wedge, \vee, 0, 1)$ is a complete lattice, (2) $(L, \otimes, 1)$ is a commutative monoid, (3) 0 is the least and 1 the greatest element, (4) the adjointness property holds for all $a, b, c \in L$, i.e., $a \otimes b \leq c \Leftrightarrow a \leq b \to c$. Then, $\otimes$ is called **multiplication**, $\to$ **residuum** and $(\otimes, \to)$ **adjoint couple**. Each of the following adjoint couples make $\mathbf{L}$ a complete residuated lattice:

Lukasiewicz: $a \otimes b := max(0, a+b-1)$ with $a \to b := min(1,\ 1-a+b)$

Gödel: $a \otimes b := min(a,b)$ with $a \to b := \begin{cases} 1, \, a \leq b \\ b, \, a \gneqq b \end{cases}$

Product: $a \otimes b := ab$ with $a \to b := \begin{cases} 1, \quad a \leq b \\ b/a, \, a \gneqq b \end{cases}$

The **hedge operator** is defined as a unary function $^* : L \to L$ which satisfies the following properties: (1) $1^* = 1$, (2) $a^* \leq a$, (3) $(a \to b)^* \leq a^* \to b^*$, and (4) $a^{**} = a^*$. Typical examples are the *identity*, i.e., for all $a \in L$ it holds that $a^* = a$, and the *globalization*, i.e., $a^* = 0$ for all $a \in L \setminus \{1\}$ and $a^* = 1$ if and only if $a = 1$.

A triple (G, M, I) is called a **formal fuzzy context** if $I : G \times M \to L$ is a fuzzy relation between the sets G and M and L is the support set of some residuated lattice. Elements from G and M are called **objects** and **attributes**, respectively. The fuzzy relation I assigns to each $g \in G$ and each $m \in M$ a truth degree $I(g,m) \in L$ to which the object g has the attribute m. For fuzzy sets $A \in L^G$ and $B \in L^M$ the **derivation operators** are defined by

$$A^{\shortmid}(m) := \bigwedge_{g \in G} (A(g)^* \to I(g,m)),\ B^{\shortmid}(g) := \bigwedge_{m \in M} (B(m) \to I(g,m)), \qquad (5)$$

for $g \in G$ and $m \in M$. Then, $A^{\shortmid}(m)$ is the truth degree of the statement "m is shared by all objects from A" and $B^{\shortmid}(g)$ is the truth degree of "g has all attributes from B". For now, we take for * the identity. It plays an important role in the computation of the stem base, as we will see later.

A **fuzzy concept** is a tuple $(A, B) \in L^G \times L^M$ such that $A^{\shortmid} = B$ and $B^{\shortmid} = A$. Then, A is called the **(fuzzy) extent** and B the **(fuzzy) intent** of (A, B). Fuzzy concepts represent maximal rectangles with truth values different from zero in the fuzzy context. The fuzzy concepts ordered by the fuzzy set inclusion form fuzzy concept lattices [9, 10]. Taking in (5) for * hedges different from the identity, we obtain the so-called fuzzy concept lattices with hedges [11].

Example 2. The fuzzy context displayed below has the object set $G = \{x, y, z\}$, the attribute set $M = \{a, b, c, d\}$ and the set of truth values is the 3-element chain $L = \{0, 0.5, 1\}$. Using the Gödel logic and the derivation operators defined in Equation 5 with the hedge * being the identity we obtain 10 fuzzy concepts. For example $(\{1, 0.5, 0\}, \{1, 1, 0, 0\})$ is a fuzzy concept. The extent contains the truth values of each object belonging to the extent, i.e., in this case x belongs fully to the set, y belongs to it with a truth value 0.5 and z does not belong to the extent. Similar affirmations can be done for the intent. Using the Lukasiewicz logic in the same setting we obtain 13 fuzzy concepts. On this set of truth values the only possible hedge operators are the identity and globalization. As one of the major roles of the hedge operators is to control the size of the fuzzy concept lattice, the

	a	b	c	d
x	1	1	0.5	0
y	1	0.5	0	1
z	1	0	0	0.5

number of fuzzy concepts will be smaller, when using in (5) a hedge different from the identity. In our example, using the globalization operator as the hedge, we obtain 6 fuzzy concepts both with the Gödel and Lukasiewicz logic. As we will see immediately, the hedges play also an important role for the attribute implications, especially for the stem base.

Fuzzy implications were studied in a series of papers by R. Belohlavek and V. Vychodil, as for example in [12, 13]. For fuzzy sets $A, B \in L^X$ the **subsethood degree** of A being a subset of B is given by $tv(A \subseteq B) = \bigwedge_{x \in X}(A(x) \to B(x))$. Let A and B be fuzzy attribute sets, then the truth value of the implication $A \to B$ is given by

$$\begin{aligned} tv(A \to B) &:= tv(\forall g \in G((\forall m \in A, (g,m) \in I) \to (\forall n \in B, (g,n) \in I))) \\ &= \bigwedge_{g \in G} (\bigwedge_{m \in M} (A(m) \to I(g,m)) \to \bigwedge_{n \in M} (B(n) \to I(g,n))) \\ &= tv(B \subseteq A''). \end{aligned}$$

Example 3. Let us go back to our fuzzy context from Example 2. Consider the Gödel logic and the derivation operators from (5) with * being the identity. Then, $b(1)'' = \{1, 0.5, 0\}' = \{1, 1, 0, 0\}$. Now, $tv(b(1) \to a(1)) = tv(\{a(1)\} \subseteq b(1)'') = 1$ and $tv(b(1) \to \{a(1), c(0.5)\}) = 0$ because $c(0.5) \notin b(1)''$. On the other hand, considering in (5) the globalization as the hedge, we obtain $b(1)'' = \{1, 0.5, 0\}' = \{1, 1, 0.5, 0\}$ and therefore $tv(b(1) \to \{a(1), c(0.5)\}) = 1$. Yet another example is $tv(b(0.5) \to b(1)) = tv(\{b(1)\} \subseteq b(0.5)'') = tv(\{b(1)\} \subseteq \{1, 0.5, 0, 0\}) = 0.5$.

Due to the large number of implications in a fuzzy and even in a crisp formal context, one is intrested in the stem base of the implications. The **stem base** is a set of implications which is non-redundant and complete. The existence and construction of the stem base for the discrete case was studied in [14], see also [6]. The problem for the fuzzy case was studied in [13]. There, the authors showed that using in (5) the globalization, the stem base of a fuzzy context is uniquely determined. Using hedges different from the globalization, a fuzzy context may have more than one stem base.

3 Fuzzy-Valued Triadic Concept Analysis

Now, we are ready to develop our fuzzy-valued triadicsetting. We will define fuzzy-valued triadic contexts, concepts and derivation operators.

For a triadic context $\mathbb{K} = (K_1, K_2, K_3, Y)$ a **dyadic-cut** (shortly **d-cut**) is defined as $c^i_\alpha := (K_j, K_k, Y^{jk}_\alpha)$, where $\{i, j, k\} = \{1, 2, 3\}$ and $\alpha \in K_i$. A d-cut is actually a special case of $\mathbb{K}^{ij}_{X_k} = (K_i, K_j, Y^{ij}_{X_k})$ for $X_k \subseteq K_k$ and $|X_k| = 1$. Each d-cut is itself a dyadic context.

Definition 1. *A **fuzzy-valued triadic context** (**f-valued triadic context**) is a quadruple $\mathbb{K} := (K_1, K_2, K_3, Y)$, where Y is a ternary fuzzy relation between the sets K_i with $i \in \{1, 2, 3\}$, i.e., $Y : K_1 \times K_2 \times K_3 \to L$ and L is the support set* *of some residuated lattice. The elements of K_1, K_2 and K_3 are called **objects**, **attributes** and **conditions**, respectively. To every triple $(k_1, k_2, k_3) \in K_1 \times K_2 \times K_3$, Y assigns a truth value $tv_{k_3}(k_1, k_2)$ to which object k_1 has attribute k_2 under condition k_3.*

The f-valued triadic context can be represented as a three-dimensional table, the entries of which are fuzzy values (see Example 4). In $\mathbb{K}$ one can interchange the roles played by the sets K_1, K_2 and K_3 requiring, for example, that Y assigns to every triple (k_2, k_3, k_1) a truth value $tv_{k_1}(k_2, k_3)$ to which attribute k_2 exists under condition k_3 having object k_1.

Definition 2. *A **fuzzy-valued triadic concept** (shortly **f-valued triconcept**) of an f-valued triadic context (K_1, K_2, K_3, Y) is a triple (A_1, A_2, A_3) with $A_1 \subseteq L^{K_1}$, $A_2 \subseteq L^{K_2}$ and $A_3 \subseteq K_3$ that is maximal with respect to component-wise set inclusion. The components A_1, A_2 and A_3 are called **(f-valued) extent**, **(f-valued) intent**, and the **modus** of (A_1, A_2, A_3), respectively. We denote by $\mathfrak{T}(\mathbb{K})$ the set of all f-valued triconcepts.*

This definition immediately implies that the d-cut $(K_1, K_2, Y^{12}_{k_3})$ is a fuzzy context for every $k_3 \in K_3$.

Example 4. We consider an f-valued triadic context with values from the 3-element chain $\{0, 0.5, 1\}$. The object set $K_1 = \{1, 2, 3, 4, 5\}$ contains 5 groups of students, the attribute set $K_2 = \{f, s, v\}$ contains 3 feelings, namely, fevered (f), serious (s), vigilant (v) and the condition set $K_3 = \{E, P, F\}$ contains the events: Doing an exam (E), giving a presentation (P) and meeting friends (F). Using the Lukasiewicz logic, we obtain 30 f-valued triconcepts and with the Gödel logic 34. For example, $(\{1, 1, 0.5, 0, 0\}, \{1, 0.5, 1\}, \{E\})$ is an f-valued triconcept meaning that while doing an exam the first two student groups and half of the third one are fevered, vigilant and moderately serious. Another example is $(\{1, 1, 1, 1, 1\}, \{0.5, 0, 0\}, \{E, P\})$ meaning that all students are moderately fevered while giving a presentation. Yet another example is $(\{1, 0, 0, 0.5, 1\}, \{1, 0, 0.5\}, \{E, P\})$ signifying that the first, the last and half of the 4-th group of students are fevered and moderately vigilant while doing an exam and giving a presentation.

	E			P			F		
	f	s	v	f	s	v	f	s	v
1	1	1	1	1	0.5	0.5	0	0.5	1
2	1	0.5	1	0.5	0	0	0	0	0.5
3	0.5	0.5	0.5	0.5	0.5	0	0	0	0.5
4	0.5	0	0.5	0.5	0.5	0.5	0	0.5	0.5
5	1	1	1	1	0.5	0.5	0	0.5	1

Fig. 2. F-valued triadic context

Lemma 1. *Every f-valued triadic context is isomorphic to a triadic context.*

Proof. According to [9], every fuzzy context is isomorphic to a formal context, namely to its double-scaled context. Every condition d-cut is a fuzzy context. By double-scaling each condition d-cut we obtain the corresponding double-scaled triadic context $\widetilde{\mathbb{K}}$ for an f-valued triadic context $\mathbb{K}$.

Formally, suppose that $\widetilde{\mathbb{K}} := (K_1^+, K_2^+, K_3, \widetilde{Y})$ is the double-scaled triadic context of $\mathbb{K} := (K_1, K_2, K_3, Y)$, the construction of which is given below. We have to show that the considered isomorphism is given by

$$\varphi : \underline{\mathfrak{T}}(\mathbb{K}) \to \underline{\mathfrak{T}}(\widetilde{\mathbb{K}}) \text{ with } \varphi(A_1, A_2, A_3) := (A_1^+, A_2^+, A_3)$$

and that the inverse map is given by

$$\psi : \underline{\mathfrak{T}}(\widetilde{\mathbb{K}}) \to \underline{\mathfrak{T}}(\mathbb{K}) \text{ with } \psi(A_1, A_2, A_3) := (A_1^{\diamond}, A_2^{\diamond}, A_3).$$

Therefore, we have to prove the following statements: For all f-valued triconcepts $(A_1, A_2, A_3), (B_1, B_2, B_3) \in \underline{\mathfrak{T}}(\mathbb{K})$ and for all $(X_1, X_2, X_3) \in \underline{\mathfrak{T}}(\widetilde{\mathbb{K}})$:

$$\begin{aligned}
&\varphi(A_1, A_2, A_3) \in \underline{\mathfrak{T}}(\widetilde{\mathbb{K}}),\\
&\psi(X_1, X_2, X_3) \in \underline{\mathfrak{T}}(\mathbb{K}),\\
&\psi\varphi(A_1, A_2, A_3) = (A_1, A_2, A_3),\\
&\varphi\psi(X_1, X_2, X_3) = (X_1, X_2, X_3),\\
&(A_1, A_2, A_3) \lesssim_i (B_1, B_2, B_3) \Leftrightarrow \varphi(A_1, A_2, A_3) \lesssim_i \varphi(B_1, B_2, B_3),
\end{aligned}$$

for all $i \in \{1, 2, 3\}$. These statements can be proven by basic properties of fuzzy sets and triadic derivation operators. Due to limitation of space, we skip the proof. □

We present the construction of $\widetilde{\mathbb{K}} := (K_1^+, K_2^+, K_3, \widetilde{Y})$, the double-scaled triadic context, for a given f-valued triadic context $\mathbb{K} = (K_1, K_2, K_3, Y)$. Let $X_i \subseteq L^{K_i}$ with $i \in \{1, 2\}$ and let L be the support set of some residuated lattice. We define

$$\begin{aligned}
X_i^+ &:= \{(k_i, \mu) \mid k_i \in K_i, \mu \in L, \mu \leq X_i(k_i)\} \subseteq K_i^* := K_i \times L,\\
X_i^{\diamond} &:= \bigvee \{\mu \mid (k_i, \mu) \in X_i\} \subseteq L^{K_i}.
\end{aligned}$$

Then, $\widetilde{Y} \subseteq K_1^+ \times K_2^+ \times K_3$ and

$$((k_1, \mu), (k_2, \lambda), k_3) \in \widetilde{Y} :\Longleftrightarrow \mu \otimes \lambda \leq tv_{k_3}(k_1, k_2) \Longleftrightarrow \mu \otimes \lambda \leq Y(k_1, k_2, k_3).$$

According to the above lemma, the f-valued triadic contexts fulfill all the properties the triadic contexts have. The f-valued triconcepts ordered by the (fuzzy) set inclusion form a complete fuzzy trilattice. Due to limitation of space we omit the proofs.

For our f-valued setting we want to obtain the corresponding $(-)^{A_k}$ and $(-)^{(i)}$ derivation operators. However, these can be defined in various ways. We distinguish between more cases for the $(-)^{(i)}$-derivation operators. In case of the $(-)^{(i)}$**-derivation operators** with $Z = X_j \times X_3 \subseteq L^{K_j} \times K_3$ and $X_i \subseteq L^{K_i}$ for $\{i, j\} = \{1, 2\}$ the situation is easy. They are defined as

$$Z \mapsto Z^{(i)} := \bigwedge_{k_j \in K_j} \{X_j(k_j) \to Y^{(i)}(k_i, (k_j, k_3)) \mid \forall k_3 \in K_3\},$$
$$X_i \mapsto X_i^{(i)} := (T_{l_3}, \{k_3 \in K_3 \mid T_{k_3} \subseteq T_{l_3}\}) \text{ for } l_3 \in K_3,$$

where $T_{l_3} := \bigwedge_{k_i \in K_i}(X_i(k_i) \to Y^{(i)}(k_i, (k_j, l_3))$ with the derivation operators from the fuzzy dyadic context $\mathbb{K}^{(i)} := (K_i, K_j \times K_3, Y^{(i)})$ and $Y^{(i)}(k_i, (k_j, k_3)) := Y(k_i, k_j, k_3)$. The $(-)^{(3)}$**-derivation operator** for $Z := X_1 \times X_2 \subseteq L^{K_1} \times L^{K_2}$ and $X_3 \subseteq K_3$ is defined by

$$Z \mapsto Z^{(3)} : = \{k_3 \in K_3 \mid k_1 \otimes k_2 \leq tv_{k_3}(k_1, k_2), \forall (k_1, k_2) \in Z\} \quad (6)$$
$$= \bigwedge_{(k_1,k_2) \in K_1 \times K_2} (Z(k_1, k_2) \to Y^{(3)}((k_1, k_2), k_3))^*, \quad (7)$$

where $Z(k_1, k_2) := X_1(k_1) \otimes X_2(k_2)$, * is the globalization in order to assure that $Z^{(3)}$ is crisp and we have the dyadic fuzzy context $\mathbb{K}^{(3)} := (K_1 \times K_2, K_3, Y^{(3)})$ with $Y^{(3)}((k_1, k_2), k_3) := Y(k_1, k_2, k_3)$. We search for the conditions which contain the maximal rectangle generated by Z.

The situation for $X_3^{(3)}$ is quite tricky. Applying the derivation operators in $\mathbb{K}^{(3)}$ for X_3, we get a truth value $l \in L$ such that $l = k_1 \otimes k_2$ instead of a tuple (k_1, k_2). To obtain such a tuple, we first have to compute the double-scaled context $\widetilde{\mathbb{K}}$. Afterwards, we use the crisp $(-)^{(3)}$-derivation operator in $\widetilde{\mathbb{K}}$ to find the components of the triconcept. Finally, we transform these into fuzzy sets as described in the construction of $\widetilde{\mathbb{K}}$. This way, we obtain the tuples $((k_1, \mu), (k_2, \nu))$ consisting of objects and attributes with their truth values instead of the truth value $k_1 \otimes k_2$.

For other approaches of fuzzy triadic data the derivation operators given in (7) and the above construction suffice for any $(-)^{(i)}$ derivation operator.

Proposition 1. *The* $(-)^{(i)}$*-derivation operators with* $i \in \{1, 2, 3\}$ *yield f-valued triconcepts.*

Proof. Suppose $X_1 \subseteq L^{K_1}, X_2 \subseteq L^{K_2}$ and $X_3 \subseteq K_3$. We have $X_1^{(1)} = (T_{l_3}, \{k_3 \mid T_{k_3} \subseteq T_{l_3}\})$, where

$$T_{l_3} = \bigwedge_{k_1 \in K_1} (X_1(k_1) \to Y^{(1)}(k_1, (k_2, l_3)))$$
$$= \bigwedge_{k_1 \in K_1} (X_1(k_1) \to Y^{12}_{l_3}(k_1, k_2)).$$

Since $\mathbb{K}^{12}_{l_3}$ is a dyadic fuzzy context, $(X_1, T_{l_3}) =: (X_1, A_2)$ is a fuzzy preconcept in $\mathbb{K}^{12}_{l_3}$, i.e., $X_1^{\shortmid} \subseteq A_2$ and $A_2^{\shortmid} \subseteq X_1$ with the derivation operators of $\mathbb{K}^{12}_{l_3}$ given by Equation (5). In particular we have $X_1^{\shortmid} \subseteq A_2$. For any $k_3 \in K_3$ if $T_{k_3} \subseteq T_{l_3}$, then (X_1, A_2) is a fuzzy preconcept also in $\mathbb{K}^{12}_{l_3 \cup k_3}$. Proceeding alike, we obtain the largest set $A_3 \subseteq K_3$ containing l_3 such that $T_{A_3} \subseteq T_{l_3}$. Then, (X_1, A_2) is a fuzzy preconcept in $\mathbb{K}^{12}_{A_3}$. So far, we obtained the last two components of the f-valued triconcept and apply on them the $(-)^{(1)}$-derivation operator to obtain the first one. Now, we have

$$\begin{aligned}(A_2 \times A_3)^{(1)} &= \bigwedge_{k_2 \in K_2} \{A_2(k_2) \to Y^1(k_1, (k_2, k_3)) \mid \forall k_3 \in A_3\} \\ &= \bigwedge_{k_2 \in K_2} (A_2(k_2) \to Y^{12}_{A_3}(k_1, k_2)),\end{aligned}$$

which is A_2 derivated in $\mathbb{K}^{12}_{A_3}$, i.e., the fist component of the triconcept, namely A_1. Since (A_1, A_2) is a fuzzy concept, it is a maximal rectangle and A_3 is the largest set containing this maximal rectangle.

We still have to check the other pair of derivation operators. Let $X_3 \subseteq K_3$, then the maximality of $X_3^{(3)} = (A_1, A_2)$ is automatically satisfied, as we obtain $X_3^{(3)}$ from the double scaled context. The maximality of $(A_1 \times A_2)^{(3)}$ follows analogously to the first case. □

As a direct consequence of this proposition, we have the following statement:

Proposition 2. *For an f-valued triconcept* (A_1, A_2, A_3) *it holds that* $A_i = (A_j \times A_k)^{(i)}$ *for* $\{i, j, k\} = \{1, 2, 3\}$ *with* $j < k$. □

For the $(-)^{A_k}$-derivation operators we also distinguish between two cases, namely when A_k is a crisp set and when it is fuzzy. When A_k is crisp, i.e., $A_k := A_3$ we proceed as follows: For $X_i \subseteq L^{K_i}$ with $i \in \{1, 2\}$ and $A_3 \subseteq K_3$ we define

$$X_1 \mapsto X_1^{A_3} := \bigwedge_{k_1 \in K_1} (X_1(k_1)^{\bullet} \to Y^{12}_{A_3}(k_1, k_2)), \tag{8}$$

$$X_2 \mapsto X_2^{A_3} := \bigwedge_{k_2 \in K_2} (X_2(k_2) \to Y^{12}_{A_3}(k_1, k_2)) \tag{9}$$

for the dyadic fuzzy context $\mathbb{K}^{12}_{A_3} := (K_1, K_2, Y^{12}_{A_3})$. where

$$\begin{aligned}&Y^{12}_{A_3} : K_1 \times K_2 \times A_3 \to L, \\ &\qquad Y^{12}_{A_3}(k_1, k_2) := \bigwedge \{tv_{k_3}(k_1, k_2) \mid \forall (k_1, k_2, k_3) \in K_1 \times K_2 \times A_3\}.\end{aligned}$$

These derivation operators are the fuzzy counterparts of the $(-)^{A_k}$-derivation operators, because A_k is crisp. In the discrete case we have $(k_i, k_j) \in Y^{i,j}_{A_k}$ if and only if for all $k_k \in A_k$ it holds that $(k_i, k_j, k_k) \in Y$. Therefore, in the fuzzy setting for $Y^{ij}_{A_3}(k_i, k_j)$, we take the minimum of the values $tv_{k_3}(k_i, k_j)$. Since $\mathbb{K}^{12}_{A_3}$ is a fuzzy context, the $(-)^{A_3}$-derivation operators form fuzzy Galois connections.

In (8) we will need the hedge $^\bullet$ for the computation of the unique stem base, however in general we take the identity for this hedge.

For the $(-)^{A_j}$-derivation operators with $\{i,j\} = \{1,2\}$ the situation is different, because A_j is a fuzzy set. In the following we discuss more possibilities to obtain these derivation operators. In such cases we are interested in the relation between K_i and K_3 for the values of A_j. This means that we are interested in just a part of the double-scaled context $\widetilde{\mathbb{K}}$, namely in $\widetilde{\mathbb{K}}_{A_j} := \bigwedge_{a_j \in A_j} (K_i^+, K_3, a_j, \widetilde{Y})$. So, we could use discrete derivation operators to compute the concepts of $\widetilde{\mathbb{K}}_{A_j}$ and afterwards transform them into fuzzy concepts. However, this is a laborious task and was presented just for a better understanding of the problem.

Another approach for the $(-)^{A_j}$**-derivation operators** is the following:

$$X_i \mapsto X_i^{A_j} := \{k_3 \in K_3 \mid k_i \otimes k_j \leq tv_{k_3}(k_i,k_j), \forall (k_i,k_j) \in X_i \times A_j\},$$
$$X_3 \mapsto X_3^{A_j} := \bigvee \{k_i \in L^{K_i} \mid k_i \otimes k_j \leq tv_{k_3}(k_i,k_j), \forall (k_3,k_j) \in X_3 \times A_j\}.$$

In this case we do not need to double-scale the context. We compute the fuzzy concept induced by X_i and A_j and check under which conditions it exists. This way we obtain $X_i^{A_j}$, i.e., the third component of the f-valued triconcept that is induced by X_i and A_j. To obtain $X_3^{A_j}$ we consider each $k_i \in L^{K_i}$ and check whether the maximal rectangle $k_i \otimes A_j$ exists under the fixed conditions of X_3. Afterwards, we take the maximum of these k_i's due to the maximality property of f-valued triconcepts. This approach is laborious, especially the computation of $X_3^{A_j}$ due to the large number of k_i's we have to check.

We will consider a more straight-forward approach by computing the fuzzy context induced by A_j. A similar approach was presented in [1]. For $X_i \in L^{K_i}$, $A_j \in L^{K_j}$ with $\{i,j\} = \{1,2\}$ and $A_3 \subseteq K_3$ we have

$$X_i \mapsto X_i^{A_j} := \bigwedge_{k_i \in K_i} (X_i(k_i)^\bullet \to Y_{A_j}^{i3}(k_i,k_3))^*, \tag{10}$$

$$X_3 \mapsto X_3^{A_j} := \bigwedge_{k_j \in K_j} (X_j(k_j) \to Y_{A_j}^{i3}(k_i,k_3)), \tag{11}$$

where $^\bullet$ and * are hedge operators. The $^\bullet$ operator is optional, as it is needed just for the computation of the stem base. It is the identity if $i = 1$. The * hedge is always a compulsory globalization in order to assure that $X_i^{A_j}$ yields a crisp set. Then, (10) and (11) are the derivation operators of the fuzzy context $(K_i, K_3, Y_{A_j}^{i3})$ where $Y_{A_j}^{i3}(k_i,k_3) := \bigwedge_{k_j \in K_j} (A_j(k_j) \to Y(k_i,k_j,k_k))$.

Considering in (10) and (11) all values for the indices, i.e., instead of $(-)^{A_j}$ we take $(-)^{A_k}$ for $\{i,j,k\} = \{1,2,3\}$, and ignoring *, these derivation operators suffice for other approaches to Fuzzy Triadic Concept Analysis. This happens due to the fact that such derivation operators yield triconcepts in which all three components are fuzzy sets.

Proposition 3. *For $\{i,j,k\} = \{1,2,3\}$ there are (fuzzy) sets $X_i \in L^{K_i}$ ($X_i \in K_i$, if $i = 3$) and $X_k \in L^{K_k}$ ($X_k \in K_k$, if $k = 3$) such that $A_j := X_i^{X_k}$, $A_i :=$*

$A_j^{X_k}$ and $A_k := (A_i \times A_j)^{(k)}$ (if $i < j$) or $A_k := (A_j \times A_i)^{(k)}$ (if $i > j$). Then, (A_1, A_2, A_3) is an f-valued triconcept denoted by $b_{ik}(X_i, X_k)$ having the smallest k-th component under all f-valued triconcepts (B_1, B_2, B_3) with the largest j-th component satisfying $X_i \subseteq B_i$ and $X_k \subseteq B_k$. Particularly, $b_{ik}(A_i, A_k) = (A_1, A_2, A_3)$ for each f-valued triconcept (A_1, A_2, A_3) of $\mathbb{K}$.

Proof. Without loss of generality we can assume $(i, j, k) = (1, 2, 3)$. Obviously, $X_1 \subseteq A_1$ and $X_3 \subseteq A_3$. We start by proving that (A_1, A_2, A_3) is indeed an f-valued triconcept. From Proposition 1 we have $A_3 = (A_1 \times A_2)$. Then, $A_2 \subseteq A_1^{(A_1 \times A_2)^{(3)}} = A_1^{A_3} \subseteq X_1^{X_3} = A_2$. Hence, $A_2 = A_1^{A_3} = (A_1 \times A_3)^{(2)}$, similarly $A_1 = (A_2 \times A_3)^{(1)}$ and together with Proposition 2 they yield an f-valued triconcept. The rest of the proof is analogous to the crisp case. Let $(B_1, B_2, B_3) \in \mathfrak{T}(\mathbb{K})$ with $X_1 \subseteq B_1$ and $X_3 \subseteq B_3$. Then, $B_2 \subseteq A_2$, because $B_2 = (B_1 \times B_3)^{(2)} = B_1^{B_3} \subseteq X_1^{X_3} = A_2$. If $B_2 = A_2$, by similar consideration as before, we obtain $B_1 \subseteq A_1$. Therefore, we have $A_3 = (A_1 \times A_2)^{(3)} \subseteq (B_1 \times B_2)^{(3)} = B_3$, finishing the first part of the proof. Now, if (A_1, A_2, A_3) is an f-valued triconcept, then $A_1^{A_3} = (A_1 \times A_3)^{(2)} = A_2$ and $A_2^{A_3} = (A_2 \times A_3)^{(1)} = A_1$. Therefore, $b_{ik}(A_1, A_3) = (A_1, A_2, A_3)$ follows by the first part of the proposition. □

4 F-valued Implications

In this section we will study f-valued implications, as generalisations of those elaborated for the discrete case in [4]. There, the authors presented various triadic implications, which are stronger than the ones developed in [15]. For a given discrete triadic context $\mathbb{K} = (K_1, K_2, K_3, Y)$ and for $R, S \subseteq K_2$ and $C \subseteq K_3$ the expression $R \xrightarrow{C} S$ was called *conditional attribute implication*. For $R, S \subseteq K_3$ and $C \subseteq K_2$ the expression $R \xrightarrow{C} S$ was called *attributional condition implication*. Implications of the form $R \to S$ with $R, S \subseteq K_2 \times K_3$ were called *attribute×condition implications*. Our main aim in the upcoming subsections is to generalise such implications to our setting.

4.1 F-valued Conditional Attribute vs. Attributional Condition Implications

In this subsection we study implications of the form: *If we are moderately vigilant during an exam, then we are also fevered* and *If we are serious during an exam, then we feel the same during our presentation.*

Definition 3. *For $R, S \subseteq L^{K_2}, C \subseteq K_3$ and globalization $^\bullet$ we call the expression $R \xrightarrow{C} S$* ***f-valued conditional attribute implication*** *and its truth value is given by*

$$
\begin{aligned}
R \xrightarrow{C} S &:= tv(\forall g \in K_1((\forall m \in R, (g, m) \times C \in Y)^\bullet \to (\forall n \in S, (g, n) \times C \in Y))) \\
&= \bigwedge_{g \in K_1} \Big(\bigwedge_{m \in K_2} (R(m) \to Y_C^{12}(g, m)) \to \bigwedge_{n \in K_2} (S(n) \to Y_C^{12}(g, n)) \Big) \\
&= tv(S \subseteq R^{CC}).
\end{aligned}
$$

Note that these implications are ordinary fuzzy implications since we are working in the fuzzy context $\mathbb{K}_C^{12}$.

Example 5. For the context given in Figure 2 we have, for example, the f-valued conditional attribute implication $s(0.5) \stackrel{E}{\rightarrow} f(1) = s(0.5) \stackrel{P}{\rightarrow} f(1) = 0.5$ and yet another is $s(0.5) \stackrel{F}{\rightarrow} f(1) = 0$. The first implication means that whenever the students are partially serious during an exam then they are also fevered. The same holds for this implication during a presentation given by the students. The implication does not hold when they are meeting their friends. In such situations the students can be serious but have a relaxed attitude.

For an f-valued triadic context $\mathbb{K}$ we denote by

$$Imp(K_2) := \{R \to S \mid R, S \in L^{K_2}\}$$

the set of all fuzzy implications on K_2. We construct the dyadic context

$$\mathfrak{C}_{imp}(\mathbb{K}) := (Imp(K_2), K_3, I)$$

where $Imp(K_2)$ is a fuzzy set, K_3 is a crisp set and $I(R \to S, c) := R \stackrel{c}{\to} S$. In order to keep the condition set crisp, we use in $\mathfrak{C}_{imp}(\mathbb{K})$ a slightly different version of the dyadic fuzzy derivation operators defined in (5), namely

$$A^{\shortmid}(m) := \bigwedge_{g \in Imp(K_2)} (A(g)^* \to I(g,m)),\ B^{\shortmid}(g) := (\bigwedge_{m \in K_3} (B(m) \to I(g,m)))^{\bullet}$$

for $A \in Imp(K_2)$, $B \in K_3$ and * is the globalization. Then, $(A, B) \in \underline{\mathfrak{B}}(\mathfrak{C}_{imp}(\mathbb{K}))$ contains in its extent all the implications that hold under all conditions of B. As in the crisp case, each extent is an implicational theory and hence, every extent has a stem base. In the concept lattice of $\mathfrak{C}_{imp}(\mathbb{K})$ the implicational theories are hierarchically ordered by the conditions under which they hold. The extent A is the set of all implications that hold in (K_1, K_2, Y_c^{12}) with $c \in C$.

The number of fuzzy implications can be very large, since we have all implications $A \to B$ with $A, B \subseteq L^{K_2}$. In the crisp case an implication either holds or not, whereas in the fuzzy case an implication holds with a given truth value, i.e., with $tv(A \to B)$. We have $tv(a \to b, c) = \bigwedge\{tv(a \to b), tv(a \to c)\}$ and $tv(a, b \to c) = \bigwedge\{tv(a \to c), tv(b \to c)\}$ for all $a, b, c \in L^{K_2}$. Hence, for the structure of $\mathfrak{C}_{imp}(\mathbb{K})$ it is enough to compute implications of the form $a \to b$ and $a(\mu) \to a(\nu)$ for all $a, b \in L^{K_2}$ with $b \neq a$ and $\mu, \nu \in L$ with $\mu \lneq \nu$. As discussed before, the other implications are infimum reducible elements in the lattice.

In accordance with the idea presented in [4] we label the concept lattice of $\mathfrak{C}_{imp}(\mathbb{K})$ as follows: The attribute labelling is done in the usual way. For the object labelling the situation is more cumbersome. Each set of implications from $Imp(K_2)$ is an extent of $\mathfrak{C}_{imp}(\mathbb{K})$ and an implicational theory, as discussed above. The object labels shall be distributed such that every extent is generated as an implicational theory by the labels attached to it and to its subconcepts. Therefore, the bottom element of the lattice will contain the stem base of all f-valued conditional attribute implications.

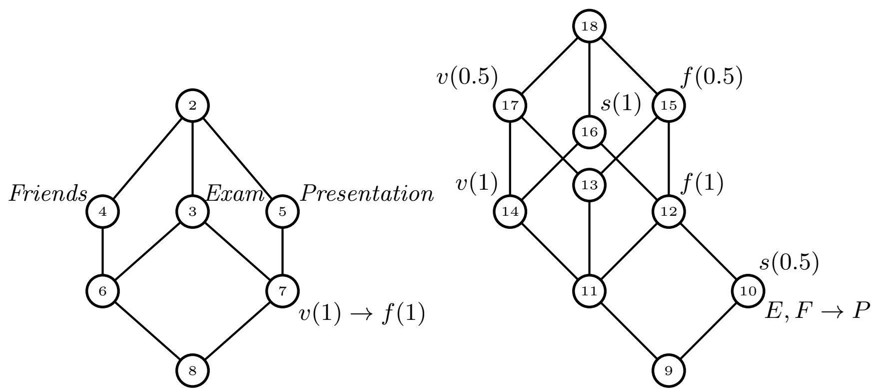

Fig. 3. Conditional attribute vs. attributional condition implications

On the left part in Figure 3 the lattice of $\mathfrak{C}_{imp}(\mathbb{K})$ is displayed. For better legibility we used just the attribute labels (the conditions) and one object label (conditional attribute implication). The implication $v(1) \to f(1)$ from the lattice means that whenever the students are vigilant in degree (truth value) 1 during an exam and presentation they are also fevered in degree 1 in these situations.

An implication $C \to D$ between the intents of $\mathfrak{C}_{imp}(\mathbb{K})$ means that if $R \stackrel{C}{\to} S$ holds, then $R \stackrel{D}{\to} S$ must hold as well. For our example the stem base of $\mathfrak{C}_{imp}(\mathbb{K})$ is $P, F \to E$. We could perform a condition attribute exploration as proposed in [4] for the discrete case, however this would go beyond the scope of this paper.

In a triadic context we may arbitrarily interchange the roles of objects, attributes and conditions. Therefore, a triadic context has a sixfold symmetry. By interchanging attributes with conditions in Definition 3, we obtain the *attributional condition implications* defined as follows:

Definition 4. *For $R, S \subseteq K_3$ and $M \subseteq L^{K_2}$ the expression*

$$R \stackrel{M}{\to} S := tv(\forall g \in K_1((\forall a \in R, g \times M \times a \in Y^*) \to (\forall b \in S, g \times M \times b \in Y^*)))$$
$$= \bigwedge_{g \in K_1} \Big(\bigwedge_{a \in K_3} (R(a) \to Y_M^{13}(g,a))^* \to \bigwedge_{b \in K_3} (S(b) \to Y_M^{13}(g,a))^* \Big),$$

*is called **f-valued attributional condition implication**, where * is the globalization.*

We use the globalization hedge operator because this time $R \stackrel{M}{\to} S$ is a crisp implication. For example, for the f-valued triadic context from Table 2 we have the attributional condition implication $P \stackrel{v(1)}{\to} E, F = 1$, meaning that students who are vigilant during a presentation are also vigilant during an exam and while meeting friends. On the other hand, $P \stackrel{f(1)}{\to} E, F = 0$ means that a student being fevered during a presentation does not imply that he/she is fevered during an exam and while spending time with friends.

In analogy to the conditional attribute implications, we can also build the context $\mathfrak{C}_{imp}(\widetilde{\mathbb{K}}) := (Imp(K_3), K_2 \times L, I)$ for the attributional condition implications. This time we have $Imp(K_3) := \{R \to S \mid R, S \in K_3\}$, i.e., all implications on K_3 and $I(R \to S, m) := R \overset{m}{\to} S$. The extents of $\mathfrak{C}_{imp}(\widetilde{\mathbb{K}})$ consist of all implications that hold in (K_1, K_3, Y_m^{13}) with $m \in K_2$. The concept lattice is displayed on the right in Figure 3. For example the implication $E, F \to P$ means that if the students during an exam and while meeting friends are (partially) fevered and (partially) serious, then they have the same feelings during their presentation.

The connection between the two classes of implications is an open question even for the discrete case and it remains open for the f-valued triadic case as well.

4.2 F-valued Attribute×Condition Implications

As presented for the discrete case, the two classes of implications studied so far are not powerful enough to express all possible kinds of implications in a triadic context. Therefore, we will generalise the so-called attribute×condition implications to our setting. These express implications of the form *If we are serious during our presentation, then we are moderately fevered during the exam.*

Definition 5. *For $R, S \subseteq L^{K_2} \times K_3$ the expression $R \to S$ is an* ***f-valued attribute×condition implication*** *and its truth value is given by*

$$\bigwedge_{g \in K_1} \Big(\bigwedge_{(m,b) \in K_2 \times K_3} (R(m,b) \to Y(g,m,b))^{\bullet} \to \bigwedge_{(n,c) \in K_2 \times K_3} (S(n,c) \to Y(g,n,c)) \Big),$$

where $^{\bullet}$ is the globalization, if we want to compute the unique stem base, otherwise the identity.

These are the attribute implications of the fuzzy context $(K_1, K_2 \times K_3, Y^{(1)})$. Their stem base is given by the stem base of the attribute implications from $(K_1, K_2 \times K_3, Y^{(1)})$.

Obviously, such implications can be easily obtained by the f-valued conditional attribute and attributional condition implications, i.e., if we have $R \overset{C}{\to} S$ for $R, S \subseteq L^{K_2}, C \subseteq K_3$, then we can compute $R \times \{c\} \to S \times \{c\}$ for all $c \in C$. Going the other way around, namely transforming the f-valued attribute×condition implications into f-valued conditional attribute and attributional condition implications, is of course also possible.

One could also be interested in f-valued object×attribute or object×condition implications. For our example this would mean *If the first group of students is fevered, then the second one is serious.*

5 Conclusion and Further Research

First, we presented a new framework for treating triadic fuzzy data. For this setting we generalised the notions of the $(-)^{A_k}$ and $(-)^{(i)}$ derivation operators, triconcepts and trilattices. We also showed how our notions can be translated into different approaches to Fuzzy Triadic Concept Analysis studied by other authors. One of our main results is the generalisation of the $(-)^{(i)}$ derivation operator for the f-valued triadic and fuzzy triadic setting, since it is absent in other works dealing with fuzzy triadic data. Second, we generalised triadic implications to our f-valued setting. These are of major importance for the development of Fuzzy and Fuzzy-Valued Triadic Concept Analysis.

Future research will focus on the connection between the different classes of f-valued triadic implications. As mentioned at the beginning, [5] is an extended version of this paper including the factorization problem. In the future we want to apply the f-valued triadic factorization to real world data.

References

1. Belohlávek, R., Osicka, P.: Triadic concept analysis of data with fuzzy attributes. In Hu, X., Lin, T.Y., Raghavan, V.V., Grzymala-Busse, J.W., Liu, Q., Broder, A.Z., eds.: GrC, IEEE Computer Society (2010) 661–665
2. Osicka, P., Konecny, J.: General approach to triadic concept analysis 116-126. In Kryszkiewicz, M., Obiedkov, S.A., eds.: Proc. CLA 2010, University of Sevilla (2010) 116–126
3. Clara, N.: Hierarchies generated for data represented by fuzzy ternary relations. In: Proceedings of the 13th WSEAS international conference on Systems, Stevens Point, Wisconsin, USA, World Scientific and Engineering Academy and Society (WSEAS) (2009) 121–126
4. Ganter, B., Obiedkov, S.A.: Implications in triadic formal contexts. In: ICCS. (2004) 186–195
5. Glodeanu, C.: Fuzzy-valued triadic concept analysis and its applications. Technical Report MATH-AL-07-2011, Technische Universitat Dresden (September 2011)
6. Ganter, B., Wille, R.: Formale Begriffsanalyse: Mathematische Grundlagen. (1996)
7. Lehmann, F., Wille, R.: A triadic approach to formal concept analysis. In Ellis, G., Levinson, R., Rich, W., Sowa, J.F., eds.: ICCS. Volume 954 of Lecture Notes in Computer Science., Springer (1995) 32–43
8. Wille, R.: The basic theorem of triadic concept analysis. Order **12** (1995) 149–158
9. Pollandt, S.: Fuzzy Begriffe. Springer Verlag, Berlin Heidelberg New York (1997)
10. Belohlávek, R.: Fuzzy Relational Systems: Foundations and Principles. Volume 20 of IFSR Int. Series on Systems Science and Engineering. Kluwer Academic/Plenum Press (2002)
11. Belohlávek, R., Vychodil, V.: Fuzzy concept lattices constrained by hedges. JACIII **11**(6) (2007) 536–545
12. Belohlávek, R., Vychodil, V.: Attribute implications in a fuzzy setting. In: ICFCA. (2006) 45–60
13. Belohlávek, R., Vychodil, V., Chlupová, M.: Implications from data with fuzzy attributes. In: AISTA 2004 in Cooperation with the IEEE Computer Society Proceedings. (2004)
14. Guigues, J.L., Duquenne, V.: Familles minimales d'implications informatives resultant d'un tableau de donnes binaires. Math. Sci. Humaines (95) (1986)
15. Biedermann, K.: A Foundation of the Theory of Trilattices. Shaker Verlag, Aachen (1988)