Vertical decomposition of a lattice using clique separators

Anne Berry, Romain Pogorelcnik, Alain Sigayret

LIMOS UMR CNRS 6158**
Ensemble Scientifique des Cézeaux Université Blaise Pascal, F-63 173 Aubière, France.
berry@isima.fr, romain.pogorelcnik@isima.fr, sigayret@isima.fr

Abstract. A concept (or Galois) lattice is built on a binary relation; this relation can be represented by a bipartite graph. We explain how we can use the graph tool of clique minimal separator decomposition to decompose some bipartite graphs into subgraphs in linear time; each subgraph corresponds to a subrelation. We show that the lattices of these subrelations easily yield the lattice of the global relation. We also illustrate how this decomposition is a tool to help displaying the lattice.

Keywords: lattice decomposition, clique separator decomposition, lattice drawing

1 Introduction

In many algorithms dealing with hard problems, a divide-and-conquer approach is helpful in practical applications. Computing the set of concepts associated with a given context (or the set of maximal rectangles associated with a binary relation) is time-consuming, as there may be an exponential number of concepts. It would be interesting to decompose the lattice into smaller sublattices. What we propose here is to decompose the relation into smaller subrelations, compute the lattice of each subrelation, and then use these lattices to reconstruct the lattice of the global relation.

For this, we use a graph decomposition, called "clique separator decomposition", introduced by Tarjan [9], and refined afterwards (see [3] for an extensive introduction to this decomposition). The general principal is roughly the following: repeatedly find a set of vertices which are pairwise adjacent (called a clique) and whose removal disconnects the graph (called a separator), then copy this clique separator into the different connected components obtained. When the decomposition is completed, a set of subgraphs is obtained, inconveniently called 'atoms': each subgraph is a maximal subgraph containing no clique separator.

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In a previous work [2], we used graph algorithms on the complement of the bipartite graph associated with the relation. In this paper, we will apply this decomposition directly to the bipartite graph itself.

It turns out upon investigation that the subgraphs obtained divide not only the graph, but in a very similar fashion divide the matrix of the relation, the set of concepts and the lattice. When the relation has a clique separator of size two, the lattice, as we will explain further on, is divided along a vertical axis by an atom and a co-atom which correspond to the two vertices of the separator. Thus not only can the concepts be computed on the subrelations, but the Hasse diagram of the lattice can be drawn better, as no edge need cross this vertical line.

Moreover, in a bipartite graph, this decomposition can be implemented with a better worse-case time complexity than in the general case, as the clique separators can be of size one (in this case they are called articulation points) or of size two. In both cases, the entire decomposition can be computed in linear time, *i.e.* in the size of the relation, thanks to the works of [6] and [3]. Although some graphs do not have a clique separator, when there is one, the decomposition is thus a useful and non-expensive pre-processing step.

The paper is organized as follows: we will first give some more preliminaries in Section 2. In Section 3, we explain how a bipartite graph is decomposed. In Section 4, we show how to reconstruct the global lattice from the concepts obtained on the subrelations. In Section 5, we discuss using vertical decomposition as a tool for layout help. Finally, in Section 6, we conclude with some general remarks.

2 Preliminaries

We will first recall essential definitions and properties.

All the graphs will be undirected and finite. For a graph $G = (V, E)$, V is the vertex set and E is the edge set. For $xy \in E, x \neq y$, x and y are said to be *adjacent*; we say that x *sees* y. A graph is *connected* if, for every pair $\{x, y\}$ of vertices, there is a path from x to y. When a graph is not connected, the maximal connected subgraphs are called the *connected components*. For $C \subset V$, $G(C)$ denotes the subgraph induced by C. In a graph $G = (V, E)$, the *neighborhood* of a vertex $x \in V$ is the set $N_G(x) = \{y \in V, x \neq y | xy \in E\}$. $N_G(x)$ is denoted $N(x)$ when there is no ambiguity.

A *clique* is set of vertices which induces a complete graph, *i.e.* with all possible edges.

A *bipartite graph* $G = (X + Y, E)$, where + stands for disjoint union, is built on two vertex sets, X and Y, with no edge between vertices of X and no edge between vertices of Y. A *maximal biclique* of a bipartite graph $G = (X + Y, E)$ is a subgraph $G(X' + Y')$ with all possible edges between the vertices of X' and the vertices of Y'.

A *relation* $\mathcal{R} \subseteq \mathcal{O} \times \mathcal{A}$ on a set $\mathcal{O}$ of objects and a set $\mathcal{A}$ of attributes is associated with a *bipartite graph* G=$(\mathcal{O} + \mathcal{A}, E)$, which we will denote $Bip(\mathcal{R})$;

thus, for $x \in \mathcal{O}$ and $y \in \mathcal{A}$, (x, y) is in $\mathcal{R}$ iff xy is an edge of G. The maximal rectangles of the relation correspond exactly to the maximal bicliques (maximal complete bipartite subgraphs) of $Bip(\mathcal{R})$ and to the elements (the concepts) of *concept lattice* $\mathcal{L}(\mathcal{R})$ associated with context $(\mathcal{O}, \mathcal{A}, \mathcal{R})$. If $O_1 \times A_1$ and $O_2 \times A_2$ are concepts of $\mathcal{L}(\mathcal{R})$ then $O_1 \times A_1 \preceq O_2 \times A_2$ iff $O_1 \subseteq O_2$ iff $A_1 \supseteq A_2$; the corresponding bicliques on vertex sets $O_1 + A_1$ and $O_2 + A_2$ of $Bip(\mathcal{R})$ are comparable the same way.

An *atom* (resp. *co-atom*) of $\mathcal{L}(\mathcal{R})$ is a concept covering the minimum element (resp. covered by the maximum element). In the bipartite graph $Bip(\mathcal{R})$, the neighborhoods are defined as follows: for $x \in \mathcal{O}$, $N(x) = \mathcal{R}(x) = \{y \in \mathcal{A} | (x, y) \in \mathcal{R}\}$, and for $x \in \mathcal{A}$, $N(x) = \mathcal{R}^{-1}(x) = \{y \in \mathcal{O} | (y, x) \in \mathcal{R}\}$.

A *separator* in a connected graph is a set of vertices, the removal of which disconnects the graph. A *clique separator* is a separator which is a clique. Clique separator decomposition [9] of a graph $G = (V, E)$ is a process which repeatedly finds a clique separator S and copies it into the connected components of $G(V - S)$. When only minimal separators are used (see [3] for extensive general definitions), the decomposition is unique and the subgraphs obtained in the end are exactly the maximal subgraphs containing no clique separator [8], [3]. In a bipartite graph, the clique minimal separators are of size one or two. A separator of size one is called a *articulation point.* A clique separator $S = \{x, y\}$ of size two is minimal if there are two components C_1 and C_2 of $G(V - S)$ such that x and y both have at least one neighbor in C_1 as well as in C_2.

3 Decomposing the bipartite graph and the relation

In the rest of the paper, we will use the bipartite graph $Bip(\mathcal{R})$ defined by a relation $\mathcal{R} \subseteq \mathcal{O} \times \mathcal{A}$. Figure 1 shows an example of a relation with the corresponding bipartite graph.

	a	b	c	d	e	f	g	h	i
1	×		×	×	×				
2		×	×			×		×	×
3	×	×				×			
4				×	×				
5							×	×	
6		×					×		

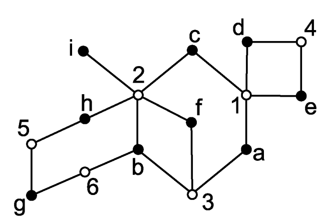


Fig. 1. A relation and the corresponding bipartite graph

In this section, we will first discuss connectivity issues, then illustrate and give our process to decompose the bipartite graph.

3.1 Decomposing the bipartite graph into connected components

When the bipartite graph $Bip(\mathcal{R})$ is not connected, our process can be applied separately (or in parallel) to each connected component. The lattice obtained is characteristic: when the top and bottom elements are removed from the Hasse diagram, the resulting diagram is a set of disjoint lattices, with a one-to-one correspondence between the connected components of $Bip(\mathcal{R})$ and the lattices obtained.

Figure 2 shows such a disconnected bipartite graph, its relation, and the corresponding lattice.

Note that trivially, if a connected component has only one vertex, this means that the corresponding row or column of the relation is empty: such a component corresponds to a lattice with only one element.

In the rest of the paper, we will consider only relations whose bipartite graph is connected.

	a	b	c	d	e	f	g
1	×		×	×			
2		×			×	×	×
3	×		×				
4	×			×			
5		×			×	×	
6		×					×
7		×			×		

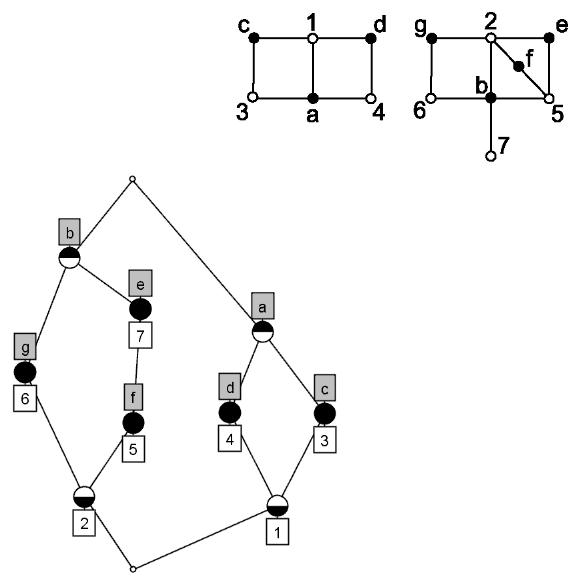


Fig. 2. A disconnected bipartite graph, its relation, and the corresponding characteristic lattice.

3.2 Illustrating the two decomposition steps

In order to make the process we use as clear as possible, we will first describe what happens when one decomposition step is applied for each of the two decompositions involved (using clique separators of size one or of size two).

It is important to understand, however, that to ensure a good (linear) time complexity, each of the two decompositions is computed globally in a single linear-time pass.

Step with an articulation point

The removal of an articulation point $\{x\}$ in a connected bipartite graph G results in components C_1, ..., C_k, which correspond to a partition $V = C_1 + ... + C_k + \{x\}$ of $Bip(\mathcal{R})$. After a decomposition step using $\{x\}$, x is preserved, with its local neighborhood, in each component, so that G is replaced by k subgraphs $G(C_1 \cup \{x\})$, ..., $G(C_k \cup \{x\})$.

Example 1. In the input graph of Figure 3, vertex 1 defines an articulation point that induces two connected components $\{2, 3, a, b, c\}$ and $\{4, d, e\}$. The decomposition step results into subgraphs $G(\{1, 2, 3, a, b, c\})$ and $G(\{1, 4, d, e\})$.

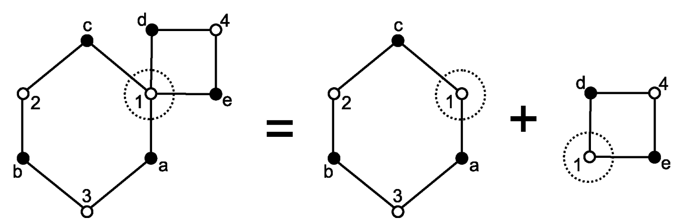


Fig. 3. Decomposition by articulation point $\{1\}$.

Step with a separator of size two

When the removal of a clique minimal separator $\{x, y\}$ in a connected bipartite graph G results into components C_1,..., C_k, corresponding to a partition $V = C_1 + ... + C_k + \{x, y\}$. The decomposition step replaces G with $G(C_1 \cup \{x, y\})$, ..., $G(C_k \cup \{x, y\})$.

Example 2. In the input graph of Figure 4, $\{2, b\}$ is a clique minimal separator of size two that induces two connected components $\{1, 2, 3, a, b, c, f\}$ and $\{2, 5, 6, b, g, h\}$.

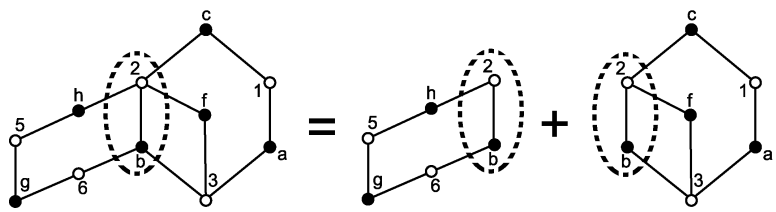


Fig. 4. Decomposition by clique separator {2,b}

3.3 Ordering the steps

A clique minimal separator of size two may include an articulation point. Thus it is important to complete the decomposition by the articulation points first, and then go on to decompose the obtained subgraphs using their clique separators of size two.

Example 3. In the input graph of Figure 5, $\{2\}$ is an articulation point included in clique minimal separator $\{2, b\}$. The decomposition will begin with $\{2\}$, inducing components $\{2, i\}$ and $\{1, 2, 3, 5, 6, a, b, c, f, g, h\}$. As there remains no articulation point in these resulting components, the second component will be then decomposed by $\{2, b\}$ into $\{1, 2, 3, a, b, c, f\}$ and $\{2, 5, 6, b, g, h\}$.

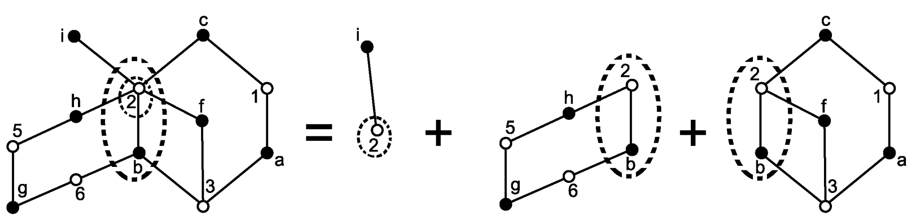


Fig. 5. Articulation point {2} is processed before clique separator {2,b}

After the bipartite graph decomposition is completed, we will obtain subgraphs with no remaining clique minimal separator, and the corresponding subrelations with their associated lattices.

Example 4. Figure 6 shows that the input graph of Figure 1 is decomposable into four bipartite subgraphs: $G_1 = G(\{1, 2, i\})$, $G_2 = G(\{2, 5, 6, b, g, h\})$, $G_3 = G(\{1, 2, 3, a, b, c, f\})$ and $G_4 = G(\{1, 4, d, e\})$.

Note that in the general case all subgraphs obtained have at least two vertices, since at least one vertex of a separator is copied into a component which has at least one vertex.

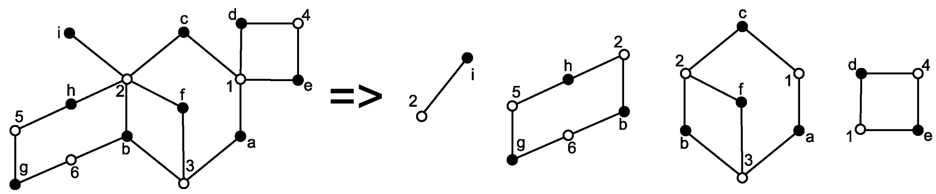


Fig. 6. Complete decomposition of a bipartite graph

3.4 The global decomposition process

To obtain the entire decomposition of a connected bipartite graph, we will thus first decompose the graph using all articulation points, and then decompose each of the subgraphs obtained using all its clique separators of size 2.

The articulation points (clique minimal separators of size one) can be found by a simple depth-first search [6], as well as the corresponding decomposition of the graph (called decomposition into biconnected components).

The search for clique separators of size two corresponds to a more complicated algorithm, described in [5]: all separators of size 2 are obtained, whether they are cliques or not.

Once this list of separators is obtained, it is easy to check which are joined by an edge. The desired decomposition can then be obtained easily.

In both cases, the set of clique separators is output.

Both algorithms run in linear time, so the global complexity is in $O(|\mathcal{R}|)$ to obtain both the set of subgraphs and the set of clique separators of the original graph.

3.5 Sub-patterns defined in the matrix

When the clique separators involved do not overlap and each defines exactly two connected components, this decomposition of the bipartite graph partitions the graph and the underlying relation. This results in a significant pattern of their binary matrix. As the components obtained are pairwise disconnected, the matrix can be reorganized in such a way that zeroes are gathered into blocks. Two components may appear as consecutive blocks, linked by a row corresponding to the articulation point that has been used to split them, or linked by one cell giving the edge between the two vertices of a size two clique minimal separator.

In the general case, this pattern can occur in only some parts of the matrix, and different patterns can be defined according to the different separators which the user chooses to represent.

Example 5. The first of the two matrices below corresponds to our running example from Figure 1 and has been reorganized, following the decomposition, which results in the second matrix. Notice how $\{1\}$ is an articulation point, so row 1 is shared by blocs $231 \times bacf$ and $14 \times de$; and how $\{2, b\}$ is a clique separator of size two, so cell $[2, b]$ is the intersection of blocs $562 \times ghb$ and $231 \times bacf$. $[2, i]$ is not integrated into the pattern, because separator $\{2, b\}$ of $Bip(\mathcal{R})$ defines 3 connected components: $\{2, 5, 6, b, g, h\}$, $\{i\}$ and $\{1, 3, 4, a, c, d, e, f\}$.

	a	b	c	d	e	f	g	h	i
1	×		×	×	×				
2		×	×			×		×	×
3	×	×				×			
4				×	×				
5							×	×	
6		×					×		

	i	g	h	b	a	c	f	d	e
5		×	×						
6		×		×					
2	×		×	×		×	×		
3				×	×		×		
1					×	×		×	×
4								×	×

We will now describe a process to organize the lines and columns of the matrix with such patterns. We will construct a meta-graph (introduced in [7] as the '*atom graph*'), whose vertices represent the subgraphs obtained by our decomposition, and where there is an edge between two such vertices if the two subgraphs which are the endpoints have a non-empty intersection which is a clique minimal separator separating the corresponding two subgraphs in the original bipartite graph. In this meta-graph, choose a chordless path; the succession of subgraphs along this path will yield a succession of rectangles in the matrix which correspond to a pattern.

Example 6. Figure 7 gives the meta-graph for our running example from Figure 1. Chordless path $(\{2, 5, 6, b, g, h\}, \{1, 2, 3, a, b, c, \}, \{1, 4, d, e\})$ was chosen for the patterning. Another possible chordless path would be $(\{2, i\}, \{1, 2, 3, a, b, c, \}, \{1, 4, d, e\})$. Finding a chordless path in a graph can be done in linear time; the meta-graph has less than $min(|\mathcal{A}|, |\mathcal{O}|)$ elements, so finding such a path costs less than $(min(|\mathcal{A}|, |\mathcal{O}|))^2$.

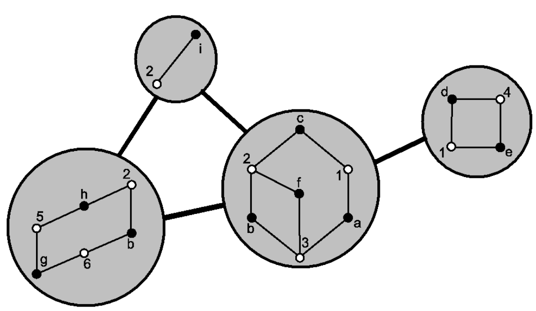


Fig. 7. Meta-graph for graph from Figure 1

3.6 Decomposing the lattice

We will now examine how the set of concepts is modified and partitioned into the subgraphs obtained. As clique minimal separators are copied in all the components induced, most of the concepts will be preserved by the decomposition. Furthermore, only initial concepts including a vertex of a clique minimal separator may be affected by the decomposition.

Definition 1. *We will say that a maximal biclique is a star maximal biclique if it contains either exactly one object or exactly one attribute. This single object or attribute will be called the center of the star.*

Lemma 1. *A star maximal biclique $\{x\} \cup N(x)$ of $Bip(\mathcal{R})$ is an atomic concept of $\mathcal{L}(\mathcal{R})$ (atom or co-atom), unless x is universal in $Bip(\mathcal{R})$. More precisely, $\{x\} \times N(x)$ is a atom if $x \in \mathcal{O}$ and $N(x) \neq \mathcal{A}$, or $N(x) \times \{x\}$ is a co-atom if $x \in \mathcal{A}$ and $N(x) \neq \mathcal{O}$.*

Proof. Let $\{x\} \cup N(x)$ be a star maximal biclique of $Bip(\mathcal{R})$. As a maximal biclique, it corresponds to a concept of $\mathcal{L}(\mathcal{R})$. Suppose the star has $x \in \mathcal{O}$ as center . As a star, it contains no other element of $\mathcal{O}$; as a biclique, it includes all $N(x) \subseteq \mathcal{A}$, and no other element of $\mathcal{A}$ by maximality. The corresponding concept is $\{x\} \times N(x)$ which is obviously the first concept from bottom to top including x. As the biclique is maximal, and as x is not universal, this concept cannot be the bottom of $\mathcal{L}(\mathcal{R})$ but only an atom. A similar proof holds for $x \in \mathcal{A}$ and co-atomicity.

We will now give the property which describes how the maximal bicliques are dispatched or modified by the decomposition. In the next Section, we will give a general theorem and its proof, from which these properties can be deduced.

Property 1. Let $G = (X + Y, E)$ be a bipartite graph, let S be a clique minimal separator of G which decomposes G into subgraphs G_1, ..., G_k. Then:

1. $\forall x \in S$, $\{x\} \cup N_G(x)$ is a star maximal biclique of G.
2. $\forall x \in S$, $\{x\} \cup N_G(x)$ is not a maximal biclique of any G_i.
3. $\forall x \in S$, $\{x\} \cup N_{G_i}(x)$ is a biclique of G_i, but it is maximal in G_i iff it is not strictly contained in any other biclique of G_i.
4. All the maximal bicliques of G which are not star bicliques with any $x \in S$ as a center are partitioned into the corresponding subgraphs.

With the help of Lemma 1, this property may be translated in terms of lattices. Given a relation $\mathcal{R}$, its associated graph G, its lattice $\mathcal{L}(\mathcal{R})$, and a decomposition step of G into some G_is by articulation point $\{x\}$:

If $x \in \mathcal{O}$ (resp. $\in \mathcal{A}$) is an articulation point of G, $\{x\} \times N_G(x)$ (resp. $N_G(x) \times \{x\}$) is a concept of $\mathcal{L}(\mathcal{R})$. After the decomposition step, in each subgraph G_i of G, either this concept becomes $\{x\} \times N_{G_i}(x)$, or this concept disappears from G_i; this latter case occurs when there is in G_i some $x' \in \mathcal{O}$, the introducer of which appears after the introducer of x in $\mathcal{L}(\mathcal{R})$, from bottom to top (resp. from top to bottom if $x, x' \in \mathcal{A}$). Every other concept will appear unchanged in exactly one lattice associated with a subgraph G_i.

The same holds for each vertex of a size two clique minimal separator.

Example 7. Figure 8 illustrates a decomposition step with articulation point $\{1\}$ using the graph from Figure 3. Concept $\{1,4\} \times \{d,e\}$ disappears from the first component $\{1,2,3,a,b,c\}$, but remains in the second component $\{1,4,d,e\}$.

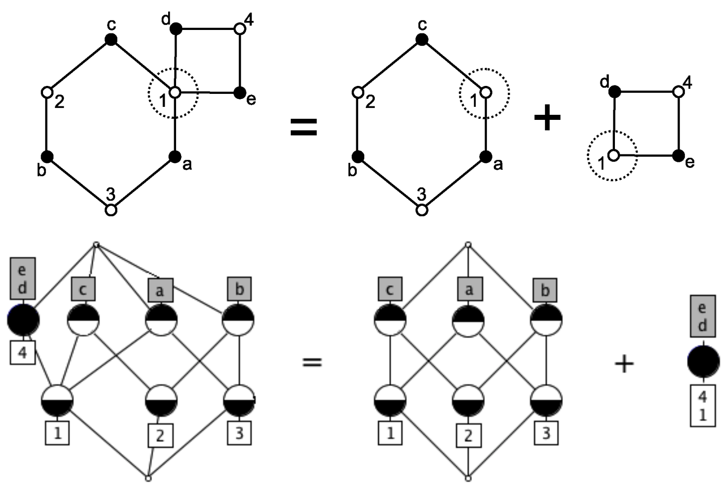


Fig. 8. Example of lattice decomposition using articulation point $\{1\}$.

Example 8. Figure 9 illustrates a decomposition step with clique separator $\{2,b\}$ using the graph from Figure 4. Concept $\{2\} \times N(2)$ is duplicated into components $\{2,5,6,b,g,h\}$ and $\{1,2,3,a,b,c,f\}$; concept $N(b) \times \{b\}$ will appear as $\{2,6\} \times \{b\}$ in the first component, but not in the second one, as $\{2,3,b\}$ is a biclique included in maximal biclique $\{2,3,b,f\}$ of G.

Remark 1. The smaller lattices obtained can not be called sublattices of the initial lattice as some of their elements may not be the same: for example, in Figure 9, $\{2\} \times \{b,c,f\}$ is an element of the third smaller lattice $\mathcal{L}(G_3)$ but is not an element of the initial lattice $\mathcal{L}(G)$.

4 Reconstructing the lattice

We will now explain how, given the subgraphs obtained by clique decomposition, as well as the corresponding subrelations and subsets of concepts, we can reconstruct the set of concepts of the global input bipartite graph. We will then go on to explain how to obtain the Hasse diagram of the reconstructed lattice.

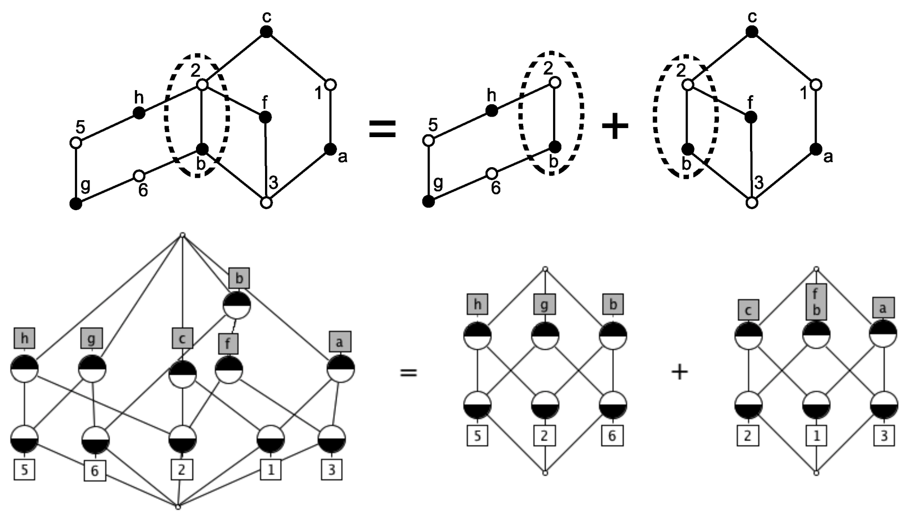


Fig. 9. Example of lattice decomposition using clique separator $\{2, b\}$.

4.1 Reconstructing the set of concepts

We will use the following Theorem, which describes the concepts of the global lattice.

Theorem 1. *Let $G = (X + Y, E)$ be a bipartite graph, let $\Sigma = \{s_1, ...s_h\}$ be the set of all the vertices which belong to a clique separator of G, let $G_1, ...G_k$ be the set of subgraphs obtained by the complete corresponding clique separator decomposition. Then:*

1. *For every $s \in \Sigma$, $\{s\} \cup N_G(s)$ is a star maximal biclique of G.*
2. *Any maximal biclique of a subgraph G_i which is not a star with a vertex of Σ as center is also a maximal biclique of G.*
3. *There are no other maximal bicliques in G: $\forall s \in \Sigma$, no other star maximal biclique of G_i with center s is a star maximal biclique of G, and these are the only maximal bicliques of some graph G_i which are not also maximal bicliques in G.*

Proof.

1. For every $s \in \Sigma$, $\{s\} \cup N_G(s)$ is a star maximal biclique of G:
   Case 1: s is an articulation point, let G_i, G_j be two graphs which s belongs to; s must be adjacent to some vertex y_i in G_i and to some vertex y_j in G_j. Suppose $\{s\} \cup N_G(s)$ is not a maximal biclique: there must be a vertex z in G which sees y_i and y_j, but then $\{s\}$ cannot separate y_i from y_j, a contradiction.
   Case 2: s is not an articulation point, let s' be a vertex of S such that

$\{s, s'\}$ is a clique separator of G, separating G_i from G_j. s must as above see some vertex y_i in G_i and some vertex y_j in G_j. Suppose $\{s\} \cup N_G(s)$ is not maximal: there must be some vertex w in G which sees all of $N_G(s)$, but w must see y_i and y_j, so $\{s, s'\}$ cannot separate G_i from G_j.

2. Let B be a non-star maximal biclique of G_i, containing $o_1, o_2 \in \mathcal{O}$ and $a_1, a_2 \in \mathcal{A}$. Suppose B is not maximal in G: there must be a vertex y in $G - B$ which augments B. Let y be in G_j, *wlog* $y \in \mathcal{A}$: y must see o_1 and o_2. Since G_i is a maximal subgraph with no clique separator, $G_i + \{y\}$ must have a clique separator. Therefore $N(y)$ must be a clique separator of this subgraph, but this is impossible, since y sees two non-adjacent vertices of G_i.
3. Any star maximal biclique B of G_i whose center is not in Σ is also a star maximal biclique of G: suppose we can augment B in G.
   Case 1: v sees an extra vertex w; $G_i + \{w\}$ contains as above a clique separator, which is impossible since $N(w) = v$ and $v \notin S$.
   Case 2: A vertex z of G_j is adjacent to all of $N(v)$: again, $G + \{z\}$ contains a clique separator, so $N(z)$ is a clique separator, but that is impossible since $N(z)$ contains at least two non-adjacent vertices.
4. For $s \in \Sigma$, no star maximal biclique of G_i is a star maximal biclique of G: let B be a star maximal biclique of G_i, with $s \in \Sigma$ as center. $s \in \Sigma$, so s belongs to some clique separator which separates G_i from some graph G_j. s must see a vertex y_j in G_j, so $B + \{y_j\}$ is a larger star including B: B cannot be maximal in G.

Example 9. We illustrate Theorem 1 using graph G from Figure 6, whose decomposition yields subgraphs G_1, ..., G_4, with $G_1 = G(\{1, 2, i\})$, $G_2 = G(\{2, 5, 6, b, g, h\})$, $G_3 = G(\{1, 2, 3, a, b, c, f\})$ and $G_4 = G(\{1, 4, d, e\})$. Finally, $\Sigma = \{1, 2, b\}$. The corresponding lattices are shown in Figure 10, and their concepts are presented in the table below. In this table, braces have been omitted; symbol $\Rightarrow$ represents a concept of the considered subgraph G_i which is identical to a concept of G (there can be only one $\Rightarrow$ per row); the other concepts of the subgraphs will not be preserved in G while recomposing.

$\mathcal{L}(G)$	$\mathcal{L}(G_1)$	$\mathcal{L}(G_2)$	$\mathcal{L}(G_3)$	$\mathcal{L}(G_4)$	star max. biclique of G ?
$1 \times acde$			$1 \times ac$		yes
$2 \times bcfhi$	$2 \times i$	$2 \times bh$	$2 \times bcf$		yes
$3 \times abf$			$\Rightarrow$		
$14 \times de$				$\Rightarrow$	
$5 \times gh$		$\Rightarrow$			
$6 \times bg$		$\Rightarrow$			
$13 \times a$			$\Rightarrow$		
$236 \times b$		$26 \times b$			yes
$12 \times c$			$\Rightarrow$		
$23 \times bf$			$\Rightarrow$		
$56 \times g$		$\Rightarrow$			
$25 \times h$		$\Rightarrow$			

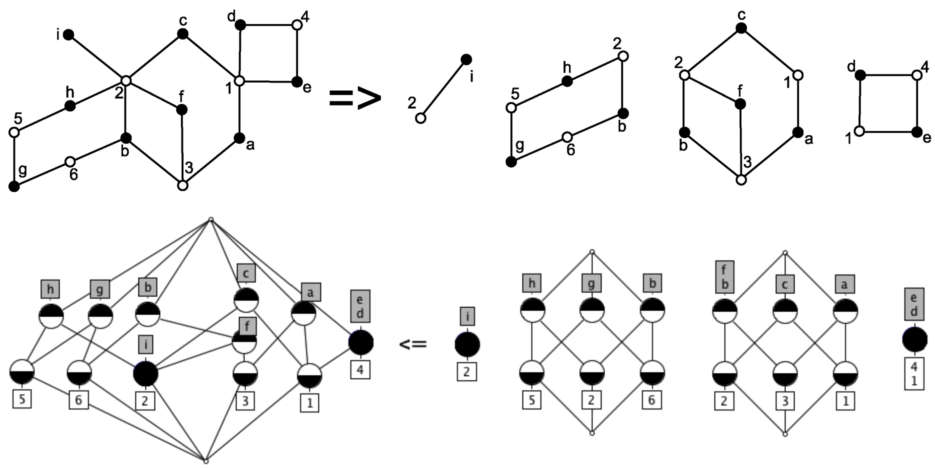


Fig. 10. Reconstruction of a lattice

According to Theorem 1, the steps to reconstruct the maximal concepts of the global lattice from the concepts of the smaller lattices are:

1. Compute Σ, the set of attributes and objects involved in a clique minimal separator. (In our example, $\Sigma = \{1, 2, b\}$.)
2. Compute the maximal star bicliques for all the elements of Σ. (In our example, we will compute star maximal bicliques $1 \times acde$, $2 \times bcfhi$ and $26 \times b$.)
3. For each smaller lattice, remove from the set of concepts the atoms or co-atoms corresponding to elements of Σ; maintain all the other concepts as concepts of the global lattice. (In our example, for $\mathcal{L}(G_3)$, we will remove $1 \times ac$ and $2 \times bcf$, and maintain $3 \times abf, 13 \times a, 12 \times c$ and $23 \times bf$ as concepts of $\mathcal{L}(G)$.)

Step 1 requires $O(|\mathcal{R}|)$ time. Step 2 can be done while computing the smaller lattices; Step 3 costs constant time per concept. Thus the overall complexity of the reconstruction is in $O(|\mathcal{R}|)$ time.

4.2 Reconstructing the edges of the Hasse diagram

According to Theorem 1, the maximal bicliques which are not star maximal bicliques with a vertex of Σ as center are preserved; therefore, the corresponding edges between the elements of the lattice are also preserved. In the process described below, we will refer to labels in the lattice as being the 'reduced' labels, such as the ones used in our lattice figures throuhout this paper.

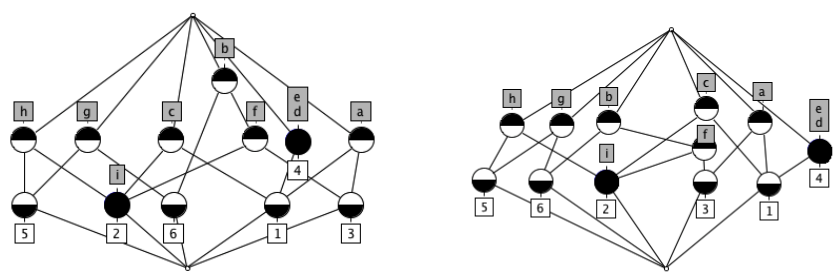
To define the edges of the Hasse diagram of lattice $\mathcal{L}(G)$, we will, for each smaller lattice $\mathcal{L}(G_i)$:

- find each atom (or co-atom) which corresponds to an element σ of Σ (such as 2 or b for $\mathcal{L}(G_3)$ in our example).
- If σ shares its label with some non-elements of Σ, remove all elements of Σ from the label. (In our example for $\mathcal{L}(G_3)$, bf becomes f). If σ does not share its label with some non-elements of Σ, remove the atom or co-atom. (In our example for $\mathcal{L}(G_3)$, remove element 2).
- Maintain the remaining edges as edges of $\mathcal{L}(G)$.
- Compute the neighborhood in $\mathcal{L}(G)$ of each atom or co-atom which corresponds to an element of Σ.

All this can be done in polynomial time: there are at most $|\mathcal{A}| + |\mathcal{O}|$ vertices in Σ, and the corresponding edges can be added in $O((|\mathcal{A}| + |\mathcal{O}|)^2|\mathcal{R}|)$.

5 Vertical decomposition as a layout tool

When there is a size two clique separator in the bipartite graph which divides the graph into two components, the concepts which are not involved in the separator can be displayed on the two sides of the separator, thus helping to minimize the number of line crossings in the Hasse diagram.



(a) (b)

Fig. 11. **(a)** Lattice constructed by Concept Explorer using the minimal intersection layout option (8 crossings). **(b)** Lattice re-drawn using the information on clique separators (5 crossings).

To illustrate this, we have used our running example with 'Concept Explorer' [1], which is a very nice and user-friendly tool for handling lattices. Notice however how clique separator $\{1, d\}$ is better displayed when put at the right extremity.

Figure 11 shows the lattice as proposed by Concept Explorer, and then redrawn with insight on the clique separators of the bipartite graph.

The same technique of course also applies when there is a succession of such clique separators.

Let us add that if moreover both lattices are planar, as discussed in [4], merging the two lattices obtained using the clique separator as central will preserve planarity.

6 Conclusion and perspectives

We have used a graph method, clique minimal separator decomposition, to provide simple tools which can help reduce the time spent computing the elements of a lattice, as well as improve the drawing of its Hasse diagram.

When there is no clique separator in the bipartite graph, it could be interesting to investigate restricting the relation to a subgraph or partial subgraph which does have one.

We leave open the question of characterizing, without computing the relation, the lattices whose underlying bipartite graph has a clique minimal separator.

Acknowledgments

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