Galois and his Connections—A retrospective on the 200th birthday of Evariste Galois

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Abstract. A frequently used tool in mathematics is what Oystein Ore called "Galois connections" (also "Galois connexions", "Galois correspondences" or "dual adjunctions"). These are pairs (φ, ψ) of maps between ordered sets in opposite direction so that $x \leq \psi(y)$ is equivalent to $y \leq \varphi(x)$. This concept looks rather simple but proves very effective. The primary gain of such "dually adjoint situations" is that the ranges of the involved maps are dually isomorphic: thus, Galois connections present two faces of the same medal.

Many concrete instances are given by what Garrett Birkhoff termed "polarities": these are nothing but Galois connections between power sets. In slightly different terminology, the fundamental observation of modern Formal Concept Analysis is that every "formal context", that is, any triple (J, M, I) where I is a relation between (the elements of) J and M, gives rise to a Galois connection (assigning to each subset of one side its "polar", "extent" or "intent" on the other side), such that the resulting two closure systems of polars are dually isomorphic; more surprising is the fact that, conversely, every dual isomorphism between two closure systems arises in a unique fashion from a relation between the underlying sets. In other words: the complete Boolean algebra of all relations between J and M is isomorphic to that of all Galois connections between $\P J$ and $\P M$, and also to that of all dual isomorphisms between closure systems on J and M, respectively.

The classical example is the Fundamental Theorem of Galois Theory, establishing a dual isomorphism between the complete lattice of all intermediate fields of a Galois extension and that of the corresponding automorphism groups, due to Richard Dedekind and Emil Artin. In contrast to that correspondence, which does not occur explicitly in Galois' succinct original articles, a few other closely related Galois connections may be discovered in his work (of course not under that name). Besides these historical forerunners, we discuss a few other highlights of mathematical theories where Galois connections enter in a convincing way through certain "orthogonality" relations, and show how the Galois approach considerably facilitates the proofs. For example, each of the following important structural isomorphisms arises from a rather simple relation on the respective ground sets:

- the dual isomorphism between the subspace lattice of a finite-dimensional linear space and the left ideal lattice of its endomorphism ring
- the duality between algebraic varieties and radical ideals
- the categorical equivalence between ordered sets and Alexandroff spaces
- the representation of complete Boolean algebras as systems of polars.