General approach to triadic concept analysis

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Abstract. Triadic concept analysis (TCA) is an extension of formal concept analysis (dyadic case) which takes into account modi (e.g. time instances, conditions, etc.) in addition to objects and attributes. Thus instead of 2-dimensional binary tables TCA concerns with 3-dimensional binary tables. In our previous work we generalized TCA to work with grades instead of binary data; in the present paper we study TCA in even more general way. In order to cover up an analogy of isotone concept-forming operators (known from dyadic case in fuzzy setting) we developed an unifying framework in which both kinds of concept-forming operators are particular cases of more general operators. We describe the unifying framework, properties of the general concept-forming operators, show their relationship to those we used in our previous work.

1 Introduction

The triadic approach to concept analysis (TCA) was introduced by Wille and Liehman in [12]. TCA is an extension of Formal Concept Analysis; it is based on a formalization of the triadic relation connecting objects, attributes, and conditions (we recall the basic notions of TCA in Section 2). In our previous work [3], we generalized TCA for graded data (fuzzy setting). The present paper generalizes TCA even more.

(Antitone) Galois connections and concept lattices of data represented by a fuzzy relation (graded data) were studied in a series of papers, see e.g. [1, 14]. An alternative approach, based on antitone Galois connections, was studied in [10, 13]. The concept lattices based on the antitone and the isotone Galois connections have distinct, natural meaning. It is well known that in the ordinary (two-valued) setting, the antitone and isotone cases are mutually reducible due to the law of double negation and that such a reducibility fails in a fuzzy setting because the law of double negation is not available in fuzzy logic. Nevertheless, a framework which enables a unifying approach to both the antitone and isotone cases was recently proposed in [5, 7], see also [10]. In this paper, inspired by this approach, we provide a unifying framework that enables us to treat the isotone and antitone cases within TCA. We provide mathematical foundations of the unifying framework, show its properties, and describe the way it covers the two types of concept-forming operators. We also show that an analogy of the basic theorem holds true.

The main inspiration for the presented work, in addition to developing a general framework which covers distinct particular cases, is the recent work in relational factor analysis [4, 5, 3, 6], in which concept lattices, both the antitone and the isotone are of crucial importance because they are optimal factors for relational matrix decompositions. In particular, triadic concept lattices were shown to play the role of the space of optimal factors for factor analysis of three-way binary data in [6]. To develop a general mathematical framework that can be used for a factor analysis of ordinal (graded) data is the main goal of this paper. Proper generalization can simplify definitions (as we demonstrate in Examples and) and open door to new extenstions – we intent to use it to generalize factor analysis of three-way graded data.

The paper is organized as follows: Section 2 recalls basic notions from (crisp) triadic concept analysis. Section 3 describes the unifying framework and its basic properties. In Section 4 we turn our attention to triadic concept-forming operators, triadic concepts. Section 5 brings an analogy to basic theorem of concept (tri)-lattices. Our conclusions and future research ideas are summarized in Section 6.

2 Preliminaries

Triadic Formal Concept Analysis This section introduces the notions needed in our paper. For further information we refer to [12, 16] (triadic FCA).

A *triadic context* is a quadruple $\langle X, Y, Z, I\rangle$ where X, Y, and Z are non-empty sets, and I is a ternary relation between X, Y, and Z, i.e. $I \subseteq X \times Y \times Z$. X, Y, and Z are interpreted as the sets of objects, attributes, and conditions, respectively; I is interpreted as the incidence relation ("to have-under relation"). That is, $\langle x, y, z\rangle \in I$ is interpreted as: object x has attribute y under condition z. In this case, we say that x, y, z (or x, z, y, or the result of listing x, y, z in any other sequence) are related by I. For convenience, a triadic context is denoted by $\langle X_1, X_2, X_3, I\rangle$.

Let $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ be a triadic context. For $\{i, j, k\} = \{1, 2, 3\}$ (i.e. $i, j, k \in \{1, 2, 3\} s.t. i \neq j \neq k$) and $C_k \subseteq X_k$, we define a dyadic context

$$\mathbf{K}^{ij}_{C_k} = \langle X_i, X_j, I^{ij}_{C_k}\rangle$$

by

$$\langle x_i, x_j\rangle \in I^{ij}_{C_k} \quad \text{iff} \quad \text{for each } x_k \in C_k : \, x_i, x_j, x_k \text{ are related by } I.$$

The concept-forming operators induced by $\mathbf{K}^{ij}_{C_k}$ are defined as follows:

$$C_i^{(ijC_k)} = \{x_j \in X_j \mid \text{for each } x_i \in C_i : \, \langle x_i, x_j\rangle \in I^{ij}_{C_k}\},$$

Operators $^{(ijC_k)}$ and $^{(jiC_k)}$ form a Galois connection between X_i and X_j [9].

A *triadic concept* of $\langle X_1, X_2, X_3, I\rangle$ is a triplet $\langle C_1, C_2, C_3\rangle$ of $C_1 \subseteq X_1$, $C_2 \subseteq X_2$, and $C_3 \subseteq X_3$, such that for every $\{i, j, k\} = \{1, 2, 3\}$ we have $C_i = C_j^{(ijC_k)}$; C_1, C_2, and C_3 are called the *extent*, *intent*, and *modus* of $\langle C_1, C_2, C_3\rangle$. The set of all triadic concepts of $\langle X_1, X_2, X_3, I\rangle$ is denoted by $\mathcal{T}(X_1, X_2, X_3, I)$ and is called the concept trilattice of $\langle X_1, X_2, X_3, I\rangle$; we refer to Section 5 where the notion of a trilattice is defined.

Fuzzy sets As a structure of truth-degrees we use a complete lattice. Given a complete lattice $\mathbf{L}$, we define the usual notions [1, 11]: an $\mathbf{L}$-set (fuzzy set, graded set) A in a universe U is a mapping $A\colon U \to L$, $A(u)$ being interpreted as "the degree to which u belongs to A".

Let $\mathbf{L}^U$ denote the collection of all $\mathbf{L}$-sets in U. The operations with $\mathbf{L}$-sets are defined componentwise. For instance, the intersection of $\mathbf{L}$-sets $A, B \in \mathbf{L}^U$ is an $\mathbf{L}$-set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$, etc. We write $A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$. Note that $\mathbf{2}$-sets and operations with $\mathbf{2}$-sets can be identified with ordinary sets and operations with ordinary sets, respectively. Binary $\mathbf{L}$-relations (binary fuzzy relations) between X and Y can be thought of as $\mathbf{L}$-sets in the universe $X \times Y$; similarly for ternary $\mathbf{L}$-relations.

3 Unifying framework

In this section we describe the structure of truth degrees we use. In our previous work we used residuated lattices as a scale of truth degrees. Our current approach differs in that we allow the fuzzy sets which constitutes triadic concepts, and the input table to have complete lattices with common support set and dual order as their scales of truth degrees. Moreover, we define operations on the structure of truth degrees that are counterparts of operations from residuated lattices. This approach is inspired by [5, 7, 10].

Let $\mathbf{L} = (L, \leq)$ be a bounded complete lattice and for $i \in \{1, 2, 3, 4\}$, $\mathbf{L}_i = (L_i, \leq_i)$ be bounded lattice with $L_i = L$ and $\leq_i$ being either $\leq$ or $\leq^{-1}$. That is, each $\mathbf{L}_i$ is either $(L, \leq)$ or $(L, \leq^{-1})$. We denote the operations on $\mathbf{L}_i$ by adding the subscript i, e. g. the operations in $\mathbf{L}_2$ are denoted by $\wedge_2, \vee_2, 0_2$, and 1_2.

We consider a ternary operation $\square : \mathbf{L}_1 \times \mathbf{L}_2 \times \mathbf{L}_3 \to \mathbf{L}_4$. We assume that $\square$ commutes with suprema in all arguments. That is, for any $a, a_j \in L_1$, $b, b_j \in L_2$, $c, c_j \in L_3$ we have

$$\begin{aligned}
\square(\bigvee\nolimits_{1\, j\in J} a_j, b, c) &= \bigvee\nolimits_{4\, j\in J} \square(a_j, b, c) \\
\square(a, \bigvee\nolimits_{2\, j\in J} b_j, c) &= \bigvee\nolimits_{4\, j\in J} \square(a, b_j, c) \qquad (1) \\
\square(a, b, \bigvee\nolimits_{3\, j\in J} c_j) &= \bigvee\nolimits_{4\, j\in J} \square(a, b, c_j)
\end{aligned}$$

Furthermore, for $i, j, k \in 1, 2, 3$ we define the operations $\Box^i : \mathbf{L}_j \times \mathbf{L}_k \times \mathbf{L}_4$ as

$$\Box^i(a_j, a_k, a_4) = \bigvee_i \{a_i \,|\, \Box(a_i, a_j, a_k) \le a_4\} \tag{2}$$

For convenience we denote $\Box(a, b, c)$ also by $\Box\{b, a, c\}$ or $\Box\{c, b, a\}$ etc., and $\Box^i(a_i, a_j, a_4)$ also by $\Box^i\{a_j, a_i, a_4\}$ or $\Box^i\{a_i, a_j, a_4\}$

Example 1. Complete residuated lattice [11] is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$); $\otimes$ and $\to$ satisfy so-called adjointness property: $a \otimes b \le c$ iff $a \le b \to c$ for each $a, b, c \in L$.

Let $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ be a complete residuated lattice and $\le$ be its order. (1) Let $\mathbf{L}_i = \mathbf{L}_j = (L, \le)$ for each $i, j \in \{1, 2, 3, 4\}$ and let $\Box(a_1, a_2, a_3) = a_1 \otimes a_2 \otimes a_3$. Then operations $\Box^i$ are defined as follows:

$$\Box^1(a_2, a_3, a_4) = (a_2 \otimes a_3) \to a_4 \tag{3}$$
$$\Box^2(a_3, a_1, a_4) = (a_3 \otimes a_1) \to a_4 \tag{4}$$
$$\Box^3(a_1, a_2, a_4) = (a_1 \otimes a_2) \to a_4 \tag{5}$$

(2) Let $\mathbf{L}_1 = \mathbf{L}_2 = \langle L, \le\rangle$ and $\mathbf{L}_3 = \mathbf{L}_4 = \langle L, \le^{-1}\rangle$ and let $\Box(a_1, a_2, a_3) = (a_1 \otimes a_2) \to a_3$. Then operations $\Box^i$ are defined as follows

$$\Box^1(a_2, a_3, a_4) = (a_4 \otimes a_2) \to a_3 \tag{6}$$
$$\Box^2(a_3, a_1, a_4) = (a_4 \otimes a_1) \to a_3 \tag{7}$$
$$\Box^3(a_1, a_2, a_4) \quad = a_1 \otimes a_2 \otimes a_4 \tag{8}$$

We show usability of both sets of operators in Example 2.

The following lemma describes basic properties of the previously defined operations that we will need in rest of the paper.

Lemma 1. *x*

$\Box$ is monotone in all arguments.
(b) $\Box^i$ are monotone in first two arguments and antitone in third argument.
(c) $\Box(a_1, a_2, \Box^3(a_1, a_2, a_4)) \le a_4$, analogous formulas hold for $\Box^1$ and $\Box^2$.
(d) $\Box^3(a_1, a_2, \Box(a_1, a_2, a_3)) \ge a_3$, analogous formulas hold for $\Box^1$ and $\Box^2$.

Proof. (a) follows directly from (1)
(b) follows directly from (2)
(c)

$$\begin{aligned} &\Box(a_1, a_2, \Box^3(a_1, a_2, a_4)) = \\ &= \Box(a_1, a_2, \bigvee_3 \{a_3 \,|\, \Box(a_1, a_2, a_3) \le a_4\}) = \\ &= \bigvee_4 \{\Box(a_1, a_2, a_3) \,|\, \Box(a_1, a_2, a_3) \le a_4\} \le a_4 \end{aligned}$$

(d)

$$\Box^3(a_1, a_2, \Box(a_1, a_2, a_3)) = \\ = \bigvee_3 \{x_3 \mid \Box(a_1, a_2, x_3) \leq \Box(a_1, a_2, a_3)\} \geq a_3$$

4 Triadic context, concept-forming operators, and concepts

In this section we develop the basic notions of the general approach to triadic concept analysis. We define the notions of L-context, concept-forming operators and triadic concepts in our setting and investigate their properties.

Triadic L-context is a quadruple $\langle X, Y, Z, I\rangle$ where X, Y, Z are non-empty sets interpreted as sets of objects, attributes, and conditions, respectively. I is a ternary L-relation between X, Y and Z, i.e.: $I : X \times Y \times Z \to \mathbf{L}_4$. For every $x \in X$, $y \in Y$, and $z \in Z$, the degree $I(x, y, z)$ in which are x,y, and z related is interpreted as the degree to which object x has attribute y under condition z. For convenience, we denote $I(x, y, z)$ also by $I\{x, y, z\}$ or $I\{x, z, y\}$ or $I\{z, x, y\}$, and the triadic L-context by $\langle X_1, X_2, X_3, I\rangle$.

L-context $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ induces three concept-forming operators. For $\{i, j, k\} = \{1, 2, 3\}$ and the sets $A_i \in L^{X_i}$ and $A_k \in L^{X_k}$, the concept-forming operator is a map: $L_i \times L_k \times L_4 \to L_j$ which assigns to A_i and A_k a fuzzy set $A_j \in L^{X_j}$ defined by

$$A_j(x_j) = \bigwedge_j{}_{\substack{x_i \in X_i\\ x_k \in X_k}} \Box^j\{A_i(x_i), A_k(x_k), I\{x_i, x_j, x_k\}\}. \tag{9}$$

In this case, the concept-forming operator is denoted by $^{(ijA_k)}$, i.e. fuzzy set A_j is denoted by $A_j = A_i^{(ijA_k)}$.

Example 2. (1) Let $\mathbf{L}_i$ and $\Box$ be as in Example 1(1). Then the concept-forming operators are as follows

$$A_i^{(ijA_k)}(x_j) = \bigwedge_{\substack{x_i \in X_i\\ x_k \in X_k}} (A_i(x_i) \otimes A_k(x_k)) \to I(x_1, x_2, x_3) \tag{10}$$

for $\{i, j, k\} \in \{1, 2, 3\}$. Note that these operators are fuzzy generalizations of those described in Section 2. These concept-forming operators also appear in [8].

(2) Let $\mathbf{L}_i$ and $\Box$ be as in Example 1(2). Then the concept-forming operators are defined as follows:

$$A_1^{(12A_3)}(x_2) = \bigwedge_{\substack{x_1 \in X_1\\ x_3 \in X_3}} (I(x_1, x_2, x_3) \otimes A_1(x_1)) \to A_3(x_3) \tag{11}$$

$$A_2^{(23A_1)}(x_3) = \bigwedge_{\substack{x_1 \in X_1\\ x_2 \in X_2}} (I(x_1, x_2, x_3) \otimes A_2(x_2)) \to A_1(x_1) \tag{12}$$

$$A_3^{(31A_2)}(x_1) = \bigvee_{\substack{x_2 \in X_2\\ x_3 \in X_3}} (I(x_1, x_2, x_3) \otimes A_1(x_1) \otimes A_3(x_3)) \tag{13}$$

Note that operators (12)–(13) are selected as a triadic counterpart to (dyadic) isotone galois connections [10]. Formulas (12)–(13) are rather complicated in comparisson with the general definition (9).

A *triadic fuzzy concept* of $\langle X_1, X_2, X_3, I\rangle$ is a triplet $\langle C_1, C_2, C_3\rangle$ consisting of fuzzy sets $C_1 \in L_1^{X_1}$, $C_2 \in L_2^{X_2}$, and $C_3 \in L_3^{X_3}$, such that for every $\{i, j, k\} = \{1, 2, 3\}$ we have $C_i = C_j^{(ijC_k)}$, $C_j = C_k^{(jkC_i)}$, and $C_k = C_i^{(ikC_j)}$. The C_1, C_2, and C_3 are called the *extent*, *intent*, and *modus* of $\langle C_1, C_2, C_3\rangle$. The set of all triadic concepts of $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ is denoted by $\mathcal{T}(X_1, X_2, X_3, I)$ and is called the *concept trilattice of* $\mathbf{K}$.

We view the triadic concepts as triplets of fuzzy sets of objects, attributes, and modi. That is, a concept applies to objects to degrees; similarly for attributes and conditions. In our setting, the scales of truth degrees in which objects belong to extent, attributes belong to intent, and conditions belongs to modus are complete lattices which consists of common support set, but they may be ordered dually.

The following lemma describes basic properties of concept-forming operators.

Lemma 2. *(a)* $A_i^{(ijC_k)} = C_k^{(kjA_i)}$
(b) if $C_k \subseteq D_k$ *and* $A_i \subseteq B_i$ *then* $B_i^{(ijD_k)} \subseteq A_i^{(ijC_k)}$
(c) $A_i \subseteq (A_i^{(ijA_k)})^{(jiA_k)}$

Proof. (a)

$$A_i^{(ijC_k)}(x_j) = \bigwedge_{j}{}_{\substack{x_i \in X_i \\ x_k \in X_k}} \Box^j(A_i(x_i), C_k(x_k), I\{x_i, x_k, x_j\}) = C_k^{(kjA_i)}(x_j)$$

(b)

$$\begin{aligned} B_i^{(ijD_k)}(x_j) &= \bigwedge_{j}{}_{\substack{x_i \in X_i \\ x_k \in X_k}} \Box^j(B_i(x_i), D_k(x_k), I\{x_i, x_k, x_j\}) \leq \\ &\leq \bigwedge_{j}{}_{\substack{x_i \in X_i \\ x_k \in X_k}} \Box^j(A_i(x_i), C_k(x_k), I\{x_i, x_k, x_j\}) = A_i^{(ijC_k)} \end{aligned}$$

(c)

$$\begin{aligned} &(A_i^{(ijA_k)})^{(jiA_k)}(x_i) = \\ &= \bigwedge_{i}{}_{\substack{x_j \in X_j \\ x_k \in X_k}} \Box^i(\bigwedge_{j}{}_{\substack{x'_i \in X_i \\ x'_k \in X_k}} \Box^j(A_i(x'_i), A_k(x'_k), I\{x'_i, x'_k, x_j\}), A_k(x_k), I\{x_i, x_j, x_k\}) \geq \\ &\geq \bigwedge_{i}{}_{\substack{x_j \in X_j \\ x_k \in X_k}} \Box^i(\Box^j(A_i(x_i), A_k(x_k), I\{x_i, x_k, x_j\}), A_k(x_k), I\{x_i, x_j, x_k\}) = \\ &= \bigwedge_{i}{}_{\substack{x_j \in X_j \\ x_k \in X_k}} \bigvee_{i} \{a_i \mid \Box(\Box^j(A_i(x_i), A_k(x_k), I\{x_i, x_k, x_j\}), x_k, a_i) \leq I\{x_i, x_k, x_j\}\} \end{aligned}$$

Lemma 1(c) yields that one of the possible values of a_i is $A_i(x_i)$. Therefore, the previous formula is greater than $\bigwedge_{i\, x_j \in X_j, x_k \in X_k} A_i(x_i) = A_i(x_i)$ which concludes the proof.

Theorem 1. *Let $\{i,j,k\} = \{1,2,3\}$. Then for all triadic fuzzy concepts $\langle A_1, A_2, A_3\rangle$ and $\langle B_1, B_2, B_3\rangle$ from $\mathcal{T}(\mathbf{K})$, if $\langle A_1, A_2, A_3\rangle \preceq_i \langle B_1, B_2, B_3\rangle$ and $\langle A_1, A_2, A_3\rangle \preceq_j \langle B_1, B_2, B_3\rangle$ then $\langle B_1, B_2, B_3\rangle \preceq_k \langle A_1, A_2, A_3\rangle$.*

Proof. We have $A_k = A_i^{(ikA_j)}$ and $B_k = B_i^{(ikB_j)}$. Since $A_i \subseteq B_i$ and $A_j \subseteq B_j$, Lemma 2 yields $B_k \subseteq A_k$.

The following theorem describes a way how to compute a triadic concept. Starting with two fuzzy sets $C_i \in L_i^{X_i}$ and $C_k \in L_k^{X_k}$ we obtain a triadic concept $\langle A_1, A_2, A_3\rangle$ by three projections using the concept-forming operators. Firstly we project C_i and C_k onto A_j, then we project A_j and C_k onto A_i, and finally we project A_i and A_j onto A_k

Theorem 2. *For $C_i \in L_i^{X_i}, C_k \in L_k^{X_k}$ with $\{i,j,k\} = \{1,2,3\}$, let $A_j = C_i^{(ijC_k)}$, $A_i = A_j^{(jiC_k)}$, and $A_k = A_i^{(ikA_j)}$. Then $\langle A_1, A_2, A_3\rangle$ is a triadic fuzzy concept $\mathfrak{b}_{ik}(C_i, C_k)$.*

Moreover, $\langle A_1, A_2, A_3\rangle$ has the smallest k-th component among all triadic fuzzy concepts $\langle B_1, B_2, B_3\rangle$ with the greatest j-th component satisfying $C_i \subseteq B_i$ and $X = C_k \subseteq B_k$. In particular, $\mathfrak{b}_{ik}(A_i, A_k) = \langle A_1, A_2, A_3\rangle$ for each triadic fuzzy concept $\langle A_1, A_2, A_3\rangle$.

Proof. By lemma 2(c) we have $C_i \subseteq A_i$ and since $A_k = A_i^{(ikA_j)} = (A_j^{(jiC_k)})^{(ikA_j)} = (C_k^{(kiA_j)})^{(ikA_j)}$ we have also $C_k \subseteq A_k$.

First, we prove that $\langle A_1, A_2, A_3\rangle$ is a triadic fuzzy concept. $A_k = A_i^{(ikA_j)}$ is satisfied by definition. Consider A_j. We have $A_j = C_i^{(ijC_k)} \supseteq A_i^{(ijA_k)}$ (Lemma 2(b)) and $A_j \subseteq (A_j^{(jkA_i)})^{(kjA_i)} = A_k^{(kjA_i)} = A_i^{(ijA_k)}$. Therefore, $A_j = A_i^{ijA_k}$. The proof for A_i is similar.

Let $\langle B_1, B_2, B_3\rangle$ be a triadic fuzzy concept with $X_i \subseteq B_i$ and $X_k \subseteq B_k$. Then $B_j = B_i^{(ijB_k)} \subseteq X_i^{(ijX_k)}$, so the maximal j-th component is A_j. Let $B_j = A_j$. Then $A_i = A_j^{(ijX_k)} \supseteq B_j^{(jiB_k)} = B_i$ and thus $A_k = A_i^{(ikA_j)} \subseteq B_i^{(ikB_j)} = B_k$.

The last assertion is easily observable from the definition of triadic fuzzy concept.

In the rest of the paper we need the following notation. For fuzzy sets $A_1 \in L_1^{X_1}$, $A_2 \in L_2^{X_2}$, and $A_3 \in L_3^{X_3}$ we denote by $A_1 \times A_2 \times A_3$ the ternary L_4-relation between X_1, X_2, and X_3 defined by $(A_1 \times A_2 \times A_3)(x_1, x_2, x_3) = \square(A_1(x_1), A_2(x_2), A_3(x_3))$.

The following lemma describes a "geometric view" on triadic fuzzy concepts, i.e. that triadic fuzzy concepts can be viewed as maximal clusters contained in the input data.

Lemma 3. *(a) If $\langle A_1, A_2, A_3\rangle \in \mathcal{T}(\mathbf{K})$ then $A_1 \times A_2 \times A_3 \subseteq I$.*

(b) If $A_1 \times A_2 \times A_3 \subseteq I$ *then there is* $\langle B_1, B_2, B_3\rangle \in \mathcal{T}(\mathbf{K})$ *such that* $A_i \subseteq B_i$ *for* $i = 1, 2, 3$.

(c) Each $\langle A_1, A_2, A_3\rangle \in \mathcal{T}(\mathbf{K})$ *is maximal w.r.t. to set inclusion, i.e. there is no* $\langle B_1, B_2, B_3\rangle \in \mathcal{T}(\mathbf{K})$ *other than* $\langle A_1, A_2, A_3\rangle$ *for which* $A_i \subseteq B_i$.

Proof. (a)

$$\begin{aligned}
&\Box(A_i(x_i), A_j(x_j), \bigwedge_{k}{}_{\substack{x_i\in X_i\\ x_k\in X_k}} (A_i(x_i), A_j(x_j), I\{x_i, x_j, x_k\}) \leq \\
&\leq \Box(A_i(x_i), A_j(x_j), \Box^k(A_i(x_i), A_j(x_j), I\{x_i, x_j, x_k\})) \leq \\
&\leq I\{x_i, x_j, x_k\}
\end{aligned}$$

(b) Let $\{i, j, k\} = \{1, 2, 3\}$ and $\mathfrak{b}_{ik}(A_i, A_k) = \langle B_1, B_2, B_3\rangle$. Due Theorem 2 we have $A_i \subseteq B_i$ and $A_k \subseteq B_k$. Moreover,

$$\begin{aligned}
&B_j(x_j) = A_i^{(ijA_k)}(x_j) = \\
&= \bigwedge_{j}{}_{\substack{x_i\in X_i\\ x_k\in X_k}} \Box^j(A_i(x_i), A_k(x_k), I\{x_i, x_k, x_j\}) \geq \\
&\geq \bigwedge_{j}{}_{\substack{x_i\in X_i\\ x_k\in X_k}} \Box^j(A_i(x_i), A_k(x_k), \Box(A_i(x_i), A_k(x_k), A_j(x_j))) = \\
&= A_j(x_j)
\end{aligned}$$

(c) Let $\langle A_1, A_2, A_3\rangle$ and $\langle B_1, B_2, B_3\rangle$ be triadic concepts with $A_i \subseteq B_i$ and $k \in \{1, 2, 3\}$ be an index such that $A_j \subset B_j$. From Theorem 1 follows that there is an index $j \in \{1, 2, 3\}$ such that $A_k = B_k$. Then having $A_j = A_i^{(ijA_k)}$ $B_j = B_i^{(ijB_k)}$ Lemma 2(c) yields $B_j \subseteq A_j$ which is a contradiction.

Theorem 3 (crisp representation). *Let* $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ *be a fuzzy triadic context and* $\mathbf{K}_{crisp} = \langle X_1 \times L_1, X_2 \times L_2, X_3 \times L_3, I_{crisp}\rangle$ *with* I_{crisp} *defined by* $((x_1, a), (x_2, b), (x_3, c)) \in I_{crisp}$ *iff* $\Box(a, b, c) \leq_4 I(x_1, x_2, x_3)$ *be a triadic context. Then* $\mathcal{T}(\mathbf{K})$ *is isomorphic to* $\mathcal{T}(\mathbf{K}_{crisp})$.

Proof. Define maps $\lfloor ... \rfloor_i : L^{X_i} \to X_i \times L$ and $\lceil ... \rceil_i : X_i \times L \to L^{X_i}$ for $i \in \{1, 2, 3\}$ as follows:

$$\lfloor A_i \rfloor_i = \{(x_i, a_i) \mid a_i \leq_i A_i(x_i)\} \tag{14}$$
$$\lceil A'_i \rceil_i \;\; = \bigvee\nolimits_i \{a_i \mid (x_i, a_i) \in A'_i\} \tag{15}$$

In what follows we skip subscripts and write just $\lfloor A_i \rfloor$ and $\lceil A'_i \rceil$ instead of $\lfloor A_i \rfloor_i$ and $\lceil A'_i \rceil_i$.

Let φ be a mapping $\varphi : \mathcal{T}(\mathbf{K}) \to \mathcal{T}(\mathbf{K}_{crisp})$ defined by $\varphi(\langle A_1, A_2, A_3\rangle) = \langle \lfloor A_1 \rfloor, \lfloor A_2 \rfloor, \lfloor A_3 \rfloor\rangle$.

We show, that $\varphi(\langle A_1, A_2, A_3\rangle) \in \mathcal{T}(\mathbf{K}_{crisp})$. We have $(x_i, b) \in (\lfloor A_j \rfloor^{(ji\lfloor A_k \rfloor)}$ iff for each $((x_j, a), (x_k, c)) \in \lfloor A_j \rfloor \times \lfloor A_k \rfloor$ it holds that $((x_i, b), (x_j, a), (x_k, c)) \in$

I_{crisp} iff for each $x_j \in X_j, x_k \in X_k$, and for each $a \leq_j A_j(x_j), b \leq_k A_k(x_k)$ it holds $\square(a,b,c) \leq_4 I\{x_i,x_j,x_k\}$ iff for each $x_j \in X_j, x_k \in X_k$ we have $\square(A_j(x_j), A_k(x_k), b) \leq_4 I\{x_i,x_j,x_k\}$ iff $b \leq_i A_i(x_i)$, therefore $(\lfloor A_j \rfloor^{(ji\lfloor A_k \rfloor)} \times \lfloor A_k \rfloor)^i = \lfloor A_i \rfloor$.

Let ψ be a mapping $\psi : \mathcal{T}(\mathbf{K}_{crisp}) \to \mathcal{T}(\mathbf{K})$ defined by $\psi(\langle A_1, A_2, A_3\rangle) = \langle \lceil A_1 \rceil, \lceil A_2 \rceil, \lceil A_3 \rceil\rangle$. We show, that $\psi(\langle A_1, A_2, A_3\rangle) \in \mathcal{T}(\mathbf{K})$.

We have $(\lceil A_j \rceil^{(ji\lceil A_k \rceil)}(x_i) = b$ iff b is the maximal degree with the property that for each $x_j \in X_j, x_k \in X_k$ it holds $\square(\lceil A_j \rceil(x_j), \lceil A_k \rceil(x_j), b) \leq_4 I(x_i,x_j,x_k)$ iff b is the maximal degree with the property that for each $a \leq_i b$ and each $x_j \in X_j, x_k \in X_k$ we have $\square(\lceil A_j \rceil(x_j), \lceil A_k \rceil(x_j), a) \leq_4 I(x_i,x_j,x_k)$ iff b is the maximal degree with the property that for each $a \leq_i b$ and each $((x_j,c),(x_k,d)) \in A_j \times A_k$ we have that $((x_i,a),(x_j,c),(x_k,d)) \in I_{crisp}$ iff b is the maximal degree with the property that for each $a \leq b$ we have $(x_i,a) \in A_j^{(jiA_k)} = A_i$. Therefore $\lceil A_j \rceil^{(ji\lceil A_k \rceil)} = \lceil A_i \rceil$.

Since $\lceil \lfloor A \rfloor \rceil = A$ for each fuzzy set A, the mappings φ and ψ are mutually inverse and φ is a bijection. Moreover, $\lfloor A \rfloor \subseteq \lfloor B \rfloor$ iff $A \subseteq_i B$ for all fuzzy sets A and B and thus φ preserves $\lesssim_1, \lesssim_2, \lesssim_3$.

5 Basic theorem

In this section, we define important structural relations on the set of triadic concepts. These relations are based on the subsethood relations on the sets of objects, attributes, and modi, and are fundamental for an understanding of the structure of the set of all triadic concepts. In the final part of this section, we prove a theorem which is a generalization of the basic theorem of triadic concept analysis [16].

Consider the following relations

$$\langle A_1, A_2, A_3\rangle \lesssim_i \langle B_1, B_2, B_3\rangle \text{ iff } A_i \subseteq B_i,$$
$$\langle A_1, A_2, A_3\rangle \eqsim_i \langle B_1, B_2, B_3\rangle \text{ iff } A_i = B_i.$$

It is easy to check that $\lesssim_i$ and $\eqsim_i$ are a quasiorder and an equivalence on $\mathcal{T}(\mathbf{K})$.

Denote by $\mathcal{T}(\mathbf{K})/\eqsim_i$ the corresponding factor set with equivalence classes denoted by $[\langle A_1, A_2, A_3\rangle]_i$. Letting

$$[\langle A_1, A_2, A_3\rangle]_i \preceq_i [\langle B_1, B_2, B_3\rangle]_i \text{ iff } \langle A_1, A_2, A_3\rangle \lesssim_i \langle B_1, B_2, B_3\rangle,$$

$\preceq_i$ is an order on $\mathcal{T}(\mathbf{K})/\eqsim_i$.

Let V be a non-empty set, and for $i \in \{1,2,3\}$ let $\lesssim_i$ be quasiorder relations on V. Then we call $(V, \lesssim_1, \lesssim_2, \lesssim_3)$ a *triordered set* if and only if it holds that $v \lesssim_i w$ and $v \lesssim_j w$ implies $w \lesssim_k v$ for $\{i,j,k\} = \{1,2,3\}$ and each $v, w \in V$ and $\sim_i \cap \sim_j \cap \sim_k$ ($\sim_i = \lesssim_i \cap \gtrsim_i$) is an identity relation. Clearly, $\sim_i = \lesssim_i \cap \gtrsim_i$ is an equivalence, and $\sim_i \cap \sim_j$ is an identity relation on V. Moreover, $\sim_i$ turns $\lesssim_i$ into an ordering on $V/\sim_i$ and so $(V/\sim_i, \lesssim_i)$ is an ordered set.

An element $v \in V$ is an *ik-bound* of (V_i, V_k), $V_i, V_k \subseteq V$, if $x \lesssim_i v$ for all $x \in V_i$ and $x \lesssim_k v$ for all $x \in V_k$. An *ik*-bound v is called an *ik-limit* of (V_i, V_k) if

$u \lesssim_j v$ for all ik-bounds of (V_1, V_2) u. In an triordered set $(V, \lesssim_1, \lesssim_2, \lesssim_3)$ there is at most one ik-limit of (V_1, V_2) v with a property $u \lesssim_k v$ for all ik-limits of (V_1, V_2) u. Then we call v an ik-join of (V_i, V_k) and denote it $\nabla_{ik}(V_i, V_k)$. The triordered set $(V, \lesssim_1, \lesssim_2, \lesssim_3)$ in which the ik-join exists for all $i \neq k$ and all pairs of subsets of V is a *complete trilattice.*

For a complete trilattice $\mathcal{V} = (V, \lesssim_1, \lesssim_2, \lesssim_3)$, an *order filter* F_i on ordered set $V/\sim_i$ is defined as a subset F_i of V with the property: $x \in F_i$ and $x \lesssim_i y$ implies $y \in F_i$ for all $x, y \in V$. We denote the set of all order filters on $V/\sim_i$ by $\mathcal{F}_i(\mathcal{V})$. A *principal filter* generated by $x \in V$ is the filter $[X)_i = \{y \in V \mid x \lesssim_i y\}$. We call a subset $\mathcal{X} \in \mathcal{F}_i(\mathcal{V})$ of filters *i-dense* with respect to $\mathcal{V}$ if each principal filter of $(V, / \sim_i)$ can be obtained as an intersection of some order filters from $\mathcal{X}$.

It is easy to see that $\mathcal{T}(\mathbf{K})$ is a triordered set. Let $\kappa_i : X_i \times L_i \to \mathcal{T}(\mathbf{K})$ be a mapping defined by $\kappa_i(x_i, b) = \{\langle A_1, A_2, A_3\rangle \in \mathcal{T}(\mathbf{K}) | A_i(x_i) \geq_i b\}$ for $i \in \{1, 2, 3\}$, $x_i \in X_i$ and $b \in L$. Since the principal filter generated by $\langle A_1, A_2, A_3\rangle$ is $[\langle A_1, A_2, A_3\rangle)_i = \cap_{x_i \in X_i} \kappa_i(x_i, A_i(x_i))$, the set $\kappa_i(X_i \times L_i)$ is i-dense. Moreover, κ_i happens to satisfy $\kappa_i(x_i, a) \subseteq \kappa_i(x_i, b)$ iff $b \leq_i a$.

Theorem 4 (basic theorem). *Let* $\mathbf{K} = (X_1, X_2, X_3, I)$ *be a fuzzy triadic context. Then* $\mathcal{T}(\mathbf{K})$ *is a complete trilattice of* $\mathbf{K}$ *for which the ik-joins are defined as follows:*

$$\nabla_{ik}(\mathcal{X}_i, \mathcal{X}_j) = \mathfrak{b}_{ik}\left(\bigcup\{A_i | \langle A_1, A_2, A_3\rangle \in \mathcal{X}_i\}, \bigcup\{A_k | \langle A_1, A_2, A_3\rangle \in \mathcal{X}_k\}\right).$$

A complete trilattice $\mathcal{V} = (V, \lesssim_1, \lesssim_2, \lesssim_3)$ *is isomorphic to* $\mathcal{T}(\mathbf{K})$ *if and only if there are mappings* $\tilde{\kappa}_i : X_i \times L_i \to \mathcal{F}_i(\mathcal{V})$, $i = 1, 2, 3$, *such that*

(a) $\tilde{\kappa}_i(X_i \times L_i)$ *is i-dense with respect to* $\mathcal{V}$,
(b) $\tilde{\kappa}_i(x_i, a) \subseteq \tilde{\kappa}_i(x_i, b)$ *iff* $b \leq_i a$,
(c) $A_1 \times A_2, \times A_3 \subseteq I \Leftrightarrow \bigcap_{i=1}^{3} \bigcap_{x_i \in X_i} \tilde{\kappa}_i(x_i, A_i(x_i)) \neq \emptyset$ *for all* $A_i \in L^{X_i}$.

Proof. The first assertion follows from Theorem 2.

From Theorem 3 we know that $\mathcal{T}(\mathbf{K})$ is isomorphic to $\mathcal{T}(\mathbf{K}_{crisp})$. To prove our assertion it suffices to show that conditions (a),(b), and (c) (for $\mathcal{T}(\mathbf{K})$) are equivalent with the conditions from Wille's original basic theorem (for $\mathcal{T}(\mathbf{K}_{crisp})$).

Consider the map $\tilde{\kappa_i}^w : (X_i \times L_i) \to \mathcal{F}_i(\mathcal{V})$ defined by $\tilde{\kappa_i}^w((x_i, a)) = \tilde{\kappa}_i(x_i, a)$. Obviously, $\tilde{\kappa_i}^w$ is i-dense iff $\tilde{\kappa}_i$ is i-dense. Furthermore, we have $A_1 \times A_2 \times A_3 \subseteq I \Leftrightarrow \lfloor A_1 \rfloor \times \lfloor A_2 \rfloor \times \lfloor A_3 \rfloor \subseteq I_{crisp}$, and since if $a \leq b$ then $\tilde{\kappa_i}^w((x_i, b)) \subseteq \tilde{\kappa_i}^w((x_i, a))$, we obtain

$$\begin{aligned}
&\cap_{i=1}^{3} \cap_{(x_i,a) \in A_i} \tilde{\kappa_i}^w((x_i, a)) \neq \emptyset \Leftrightarrow \\
\Leftrightarrow &\cap_{i=1}^{3} \cap_{x_i \in X_i} \tilde{\kappa_i}^w(x_i, \vee\{c \mid (x_i, c) \in A_i\}) \neq \emptyset \Leftrightarrow \\
\Leftrightarrow &\cap_{i=1}^{3} \cap_{x_i \in X_i} \tilde{\kappa}_i(x_i, A_i(x_i)) \neq \emptyset
\end{aligned}$$

This concludes the proof.

6 Conclusion

We presented how foundations of triadic concept analysis can be developed in a very general way. We showed that the previously studied cases of fuzzy TCA, namely the TCA with isotone and TCA with antitone concept-forming operators, are just particular cases of a more general approach. We provided definitions of basic notions, described properties of concept-forming operators and triadic concepts, and proved the analogy of basic theorem of TCA using crisp representation of triadic concepts.

Our future research topics on general approach to TCA include:

- Investigation of attribute implications in the unifying framework we have developed. At the current moment we study attribute implications in a unifying framework for dyadic case.
- Generalization of the unifying framework assuming supports of the lattices L_i to be different.

References

1. Belohlavek R.: *Fuzzy Relational Systems: Foundations and Principles.* Kluwer, Academic/Plenum Publishers, New York, 2002.
2. Belohlavek R.: Concept lattices and order in fuzzy logic. *Annals of Pure and Applied Logic* **128**(1–3)(2004), 277–298.
3. Belohlavek R., Vychodil V.: Factor analysis of incidence data via novel decomposition of matrices. *Lecture Notes in Artificial Intelligence* **5548**(2009), 83-97.
4. Belohlavek R.: Optimal triangular decompositions of matrices with entries from residuated lattices. Int. J. of Approximate Reasoning **50**(8)(2009), 1250–1258.
5. Belohlavek R.: Optimal decompositions of matrices with entries from residuated lattices. Conditionally accepted to J. Logic and Computation.
6. Belohlavek R., Vychodil V.: Optimal factorization of three-way binary data. Proc. of The 2010 IEEE International Conference on Granular Computing, 2010, San Jose.
7. Belohlavek R.: Sup-t-norm and inf-residuum are one type of relational product: Unifying framework and consequences. *Preprint to be submitted.*
8. Belohlavek R., Osicka P.: Triadic concept analysis of data with fuzzy attributes. Proc. of The 2010 IEEE International Conference on Granular Computing, 2010, San Jose.
9. Ganter B., Wille R.: *Formal Concept Analysis. Mathematical Foundations.* Springer, Berlin, 1999.
10. Georgescu G., Popescu A.: Non-dual fuzzy connections. *Archive for Mathematical Logic* **43** (2004).
11. Hájek P.: *Metamathematics of Fuzzy Logic.* Kluwer, Dordrecht, 1998.
12. Lehmann F., Wille R.: A triadic approach to formal concept analysis. *Lecture Notes in Computer Science* **954**(1995), 32–43.
13. Popescu A.: A general approach to fuzzy concepts. *Math. Log. Quart.* **50**(2004), 1-17.
14. Pollandt S.: *Fuzzy Begriffe.* Springer-Verlag, Berlin/Heidelberg, 1997.
15. Ward M., Dilworth R. P.: Residuated lattices. *Trans. Amer. Math. Soc.* **45** (1939), 335–354.
16. Wille R.: The basic theorem of triadic concept analysis. *Order* **12**(1995), 149–158.
17. Zadeh L. A.: Fuzzy sets. *Inf. Control* **8**(1965), 338–353.