L-Fuzzy Concepts and linguistic variables in knowledge acquisition processes.*

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Abstract. In this work, we analyze how the linguistic labels of a linguistic variable can be a useful tool in the L-Fuzzy Concept Theory. In concrete, we study the L-Fuzzy concepts obtained from a departure set represented by means of these linguistic labels applied to the set of objects or attributes.
We also illustrate the results by means of an example.

1 Introduction.

The Formal Concept Analysis developed by Wille ([18]) tries to extract some information from a binary table that represents a formal context (X, Y, R) with X and Y two finite sets (of objects and attributes, respectively) and $R \subseteq X \times Y$. This information is obtained by means of the formal concepts which are pairs (A, B) with $A \subseteq X$, $B \subseteq Y$ fulfilling $A^* = B$ and $B^* = A$ (where $*$ is the derivation operator which associates to each object set A the set B of the attributes related to A, and vice versa). A is the extension and B the intension of the concept.

The set of the concepts derived from a context (X, Y, R) is a complete lattice and it is usually represented by a line diagram.

In some previous works ([8],[9]) we defined the L-Fuzzy context (L, X, Y, R), with L a complete lattice, X and Y the sets of objects and attributes respectively and $R \in L^{X\times Y}$ an L-Fuzzy relation between the objects and the attributes, as an extension to the fuzzy case of the Wille's formal contexts when the relation between the objects and the attributes that we want to study takes values in a complete lattice L. When we work with these L-Fuzzy contexts we use the

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derivation operators 1 and 2 defined by: For every $A \in L^X, B \in L^Y$

$$A_1(y) = \inf_{x \in X}\{I(A(x), R(x,y))\}, \quad B_2(x) = \inf_{y \in Y}\{I(B(y), R(x,y))\}$$

where I is a fuzzy implication defined in $(L, \leq)$, $I : L \times L \longrightarrow L$, which is decreasing in its first argument, and, A_1 represents, in a fuzzy way, the attributes related to the objects of A and B_2 the objects related to the attributes of B.

The information of the context is visualized by means of the L-Fuzzy concepts which are pairs $(A, A_1) \in (L^X, L^Y)$ with $A \in \text{fix}(\varphi)$ the set of fixed points of the operator φ, being this one defined by the derivation operators 1 and 2 mentioned above as $\varphi(A) = (A_1)_2 = A_{12}$. These pairs, whose first and second components are the extension and the intension respectively, represent, in a vague way, the set of objects that share some attributes.

The set $\mathcal{L} = \{(A, A_1) : A \in \text{fix}(\varphi)\}$ with the order relation $\leq$ defined as:

$$(A, A_1), (C, C_1) \in \mathcal{L}, \quad (A, A_1) \leq (C, C_1) \text{ if } A \leq C$$

(or equiv. $C_1 \leq A_1$) is a complete lattice that is said to be the L-Fuzzy concept lattice ([8],[9]).

Other extensions of the Formal Concept Analysis to the Fuzzy area are in [19], [17], [6], [13], [15], [16] and [12].

2 Obtaining the closest L-Fuzzy Concept to the departure set

The process to obtain the closest L-Fuzzy concept to a departure set $A \in L^X$ that represents our interest of study begins with the calculus of the closest fixed point of φ to A described in the previous section.

In the Formal Concept Analysis and when we use a residuated implication, this is an easy process since φ is a closure operator and, as $A^* = A^{***}$, then we only have to apply twice the derivation operator $*$ to obtain the fixed point and the associate L-Fuzzy concept.

That is, if $A \subseteq X$, then (A^{**}, A^*) is the formal concept obtained from A.

More arduous is the case of using a non residuated implication. In [9], a method to obtain this fixed points by means of a calculation process using the implication of Kleene-Dienes and the operators of Cousot [11] was proposed:

For every $A \in L^X$, the L-Fuzzy sets $\text{luis}(\varphi) \circ \text{llis}(f_2)(A)$ and $\text{llis}(\varphi) \circ \text{luis}(f_1)(A)$ are fixed points of φ verifying

$$\text{luis}(\varphi) \circ \text{llis}(f_2)(A) \leq \text{llis}(\varphi) \circ \text{luis}(f_1)(A),$$

where $\text{luis} f_1(d) = \lim \sup(d, f_1(d), {f_1}^2(d), {f_1}^3(d) \ldots)$ is the limit of an stationary upper iteration sequence for f_1 starting with d and $\text{llis}(f_2)(d) = \lim \inf(d, f_2(d), {f_2}^2(d), {f_2}^3(d) \ldots)$, the limit of an stationary lower iteration sequence for f_2 starting with d. Also, we have that $f_1(d) = d \vee \varphi(d)$ and $f_2(d) = d \wedge \varphi(d)$.

Moreover, these fixed points are greater than or equal to any fixed point of φ less than or equal to A, and less than or equal to any fixed point of φ greater than or equal to A. (Many times both fixed points are coincident.)

For any of the obtained fixed points $\hat{A}$ we calculate the closest L-Fuzzy concept $(\hat{A}, \hat{A}_1)$ to the departure set A.

The use of a non residuated implication operator complicates the process to obtain the associated concept. For this reason, most of times a residuated implication is used. For example, the Lukasiewicz one.

A study of the obtained results using different implications is in [10].

This process can also be applied to a set of attributes instead of objects.

Now, we will see an example where the L-Fuzzy concepts derived from a departure set are showed. In all the examples of this paper, we will use the lattice $L = \{0, 0.1, 0.2, \ldots 0.9, 1\}$ and the Lukasiewicz implication operator I.

Example 1 *Let (L, X, Y, R) be an L-Fuzzy context, where $X = \{x_1, x_2, x_3, x_4, x_5\}$, $Y = \{y_1, y_2, y_3, y_4\}$ and the L-Fuzzy relation R is represented by Table 1.*

Table 1. Relation R.

R	y_1	y_2	y_3	y_4
x_1	1	1	0.1	1
x_2	0.9	0.1	0	0
x_3	0.1	1	0.9	0.9
x_4	0	0.1	1	0.1
x_5	0.8	0.2	1	0

In this case, we want to study for which attributes, the membership degree of the objects x_1 and x_3 is high. To do this, we take a set $A \in L^X$ that represents the situation to analyze: $A = \{(x_1/1, x_2/0, x_3/1, x_4/0, x_5/0)\}$, obtaining the L-Fuzzy concept:

$$\{(x_1/1, x_2/0.1, x_3/1, x_4/0.1, x_5/0.1), (y_1/0.1, y_2/1, y_3/0.1, y_4/0.9)\}$$

In order to interpret the meaning of this L-Fuzzy concept, we will focus on those objects and attributes whose membership degrees stand out from the rest. In this case, we say that y_2 and y_4 are the attributes that have high values of x_1 and x_3.

However, this method not always provides satisfactory results. For example, if we want to see what objects share the attributes y_2 and y_4 but, in addition, do not have the attributes y_1 and y_3, the previous method does not give us a good result, since if we take the L-Fuzzy set $B = \{(y_1/0, y_2/1, y_3/0, y_4/1)\}$, then we obtain the L-Fuzzy concept:

$$\{(x_1/1, x_2/0, x_3/0.9, x_4/0.1, x_5/0), (y_1/0.2, y_2/1, y_3/0.1, y_4/1)\}$$

that it would be interpreted saying that x_1 and x_3 verify the required conditions. Nevertheless, x_1 does not have low values in y_1 and x_3 does not have low values in y_3.

That is, this process goes well for high values of objects or attributes, but it does not for the low ones. This same problem can be seen in the example proposed by Pollandt in [17] relating to the weather throughout one week.

Furthermore, sometimes we will be interested not only in studying high or low values of objects or attributes but also in other ones: medium, medium-high, very low etc. As we will see in the following section, we can use linguistic variables to do this.

3 Using linguistic variables to represent departure sets and calculate their associated L-Fuzzy concepts

3.1 Linguistic variables

We begin by summarizing some well-known definitions of fuzzy logic.

A *fuzzy number* [20] is a normal and convex fuzzy set. There are many kinds of fuzzy numbers, e.g. triangle, trapezoid, S-shaped, bell etc. These fuzzy numbers characterize the linguistic variables that appear next.

Taking the definition of Zadeh [20]: By a *linguistic variable* we mean a variable whose values are words or sentences instead of numbers and that is characterized by a tuple $(V, T(V), [0,1], G, M)$ where V is the name of the variable, $T(V)$ is the set of linguistic labels or values, $[0,1]$ is the Universe of discourse, G is a syntactic rule which generates the values of $T(V)$ and M is the semantic rule which assigns to each linguistic value $t \in T(V)$ its meaning $M(t)$.
The meaning of a linguistic label t is characterized by a *compatibility function* $c_t : [0,1] \to [0,1]$ which assigns its compatibility with $[0,1]$ to every t.

We will now consider linguistic variables defined in the Universal set $[0,1]$ where the meaning of the label $M(t)$ is represented by a truncated symmetrical trapezoidal fuzzy number. In concrete, we use those represented in Fig. 1 (the values a and b define the interval where $c_t(x) = 1$):

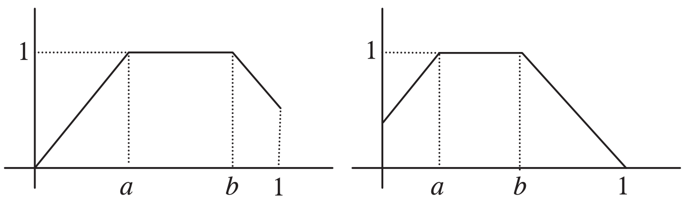

Fig. 1. Fuzzy sets assigned to labels.

Observe that these truncated trapezoidal numbers are the restriction to the interval [0,1] of the original ones defined in $\mathbb{R}$.
Notation: We denote x_t to the compatibility of the value $x \in [0,1]$ with label t.

Then for every $x \in [0,1]$:

$$c_t(x) = x_t = \begin{cases} 1 + m(x-a) & \text{if } x \leq a \\ 1 & \text{if } a \leq x \leq b \\ 1 + m(b-x) & \text{if } x \geq b \end{cases} \quad \text{where } m = \min\left\{\frac{1}{a}, \frac{1}{1-b}\right\}$$

These two values, $a, b \in [0,1]$, are the assigned to the label $t \in T(V)$ in its definition.

On the other hand, the label set $T(V)$ have to cover the whole [0,1] so that, for every $x \in [0,1]$, a only label $t \in T(V)$ exists such that $x_t = 1$. In other case, we have two different definitions from the transformed values of the relation R.

We also have proved some results about interval-valued linguistic variables in [5].

In the next section, we will see how these linguistic labels can be used to solve the problem explained at the end of Section 2: we will represent the situation that we want to study by means of a set of pairs using these linguistic labels.

3.2 *L*-Fuzzy contexts associated with labels of a linguistic variable

Associated with every label $t \in T(V)$ of a linguistic variable, we can create an L-Fuzzy context that will be used to obtain the closest L-Fuzzy concept to a departure set:

Definition 1 *Let (L, X, Y, R) be an L-Fuzzy context and let $T(V)$ be the linguistic labels set assigned to variable V. For every label $t \in T(V)$, we can create a new L-Fuzzy context (L, X, Y, R_t) where X and Y are the object and attribute sets of the original context and relation R_t is defined as follows:*

$$R_t(x_i, y_j) = R(x_i, y_j)_t, \forall x_i \in X, \forall y_j \in Y$$

and measures the compatibility of $R(x_i, y_j)$ with label t. The defined context is said to be the t-labeled L-Fuzzy context.

As corollary of the next proposition it is not difficult to prove some properties of this new context:

Proposition 1 *Let (L, X, Y, R) be an L-Fuzzy context and let I be a residuated implication operator. If we take the basic point $A \in L^X$, that is:*

$$A(x) = \begin{cases} 1 & \textit{if } x = x_i \\ 0 & \textit{otherwise} \end{cases}$$

then the intension of the L-Fuzzy concept obtained taking A as a departure point is $A_1(y) = R_t(x_i, y), \forall y \in Y$, that is, it is coincident with the row x_i of the relation R.
Moreover, the extension verifies that $A_{12}(x_i) = 1$.

In the same way, if we take as a departure point the set $B \in L^Y$*:*

$$B(y) = \begin{cases} 1 & \textit{if } y = y_j \\ 0 & \textit{otherwise} \end{cases}$$

then the extension of the obtained L-Fuzzy concept is $B_2(x) = R(x, y_j), \forall x \in X$*. That is, the values of the column* y_j *of* R*.*
Moreover, $B_{21}(y_j) = 1$ *holds.*

Proof: We take an object set as the departure point.

Let $A \in L^X$ be a basic point,

$$A(x) = \begin{cases} 1 & \text{if } x = x_i \\ 0 & \text{otherwise} \end{cases}$$

We calculate the L-Fuzzy concept derived from A in the L-Fuzzy context (L, X, Y, R) :

We can apply the derivation operator and we obtain the intension of the L-Fuzzy concept:

$$A_1(y) = \inf_{x \in X} \{I(A(x), R(x, y))\}, \forall y \in Y.$$

Since the implication operator is residuated, hence $\forall z \in [0, 1]$, $I(0, z) = 1$ and $I(1, z) = z$ holds. Then

$$A_1(y) = I(A(x_i), R(x_i, y)) = R(x_i, y), \forall y \in Y.$$

On the other hand, with respect to the extension of the L-Fuzzy concept:

$$A_{12}(x) = \inf_{y \in Y} \{I(A_1(y), R(x, y))\} = \inf_{y \in Y} \{I(R(x_i, y), R(x, y))\}.$$

And, as all the residuated implications verify $\forall z \in [0, 1]$, $I(z, z) = 1$, we can say that $A_{12}(x_i) = 1$.

□

The proof from a set of attributes is analogous.

Corollary 1 *As* $R_t(x_i, y_j) = R(x_i, y_j)_t$*, the intension (or extension) of the L-Fuzzy concept obtained from a departure set associated with a basic point created from an object (or attribute) in the t-labeled L-Fuzzy context* (L, X, Y, R_t)*, is coincident with the application of label* t *to the intension (or extension) of the L-Fuzzy concept obtained in the original L-Fuzzy context* (L, X, Y, R)*.*

Corollary 2 *Let* (L, X, Y, R_t) *be the t-labeled L-Fuzzy context, with* $t \in T(V)$*. If exists* $x_i \in X$ *such that* $R_t(x_i, y_j) = 1$*, for some* $y_j \in Y$*, and we have* $A \in L^X$*:*

$$A(x) = \begin{cases} 1 & \textit{if } x = x_i \\ 0 & \textit{otherwise} \end{cases}$$

Then the L-Fuzzy concept of the t-labeled L-Fuzzy context obtained taking A as a departure set verifies that $A_1(y_j) = 1$ *and* $A_{12}(x_i) = 1$.

Analogous, If exists $y_j \in Y$ *such that* $R_t(x_i, y_j) = 1$, *for some* $x_i \in X$, *and we have the set* $B \in L^Y$:

$$B(y) = \begin{cases} 1 & \text{if } y = y_j \\ 0 & \text{otherwise} \end{cases}$$

Then the L-Fuzzy concept of the t-labeled L-Fuzzy context obtained taking B as a departure set verifies that $B_2(x_i) = 1$ *y* $B_{21}(y_j) = 1$.

That is, the elements x_i *and* y_j *are outstanding elements in the obtained L-Fuzzy concepts.*

Now, we are going to see an illustrative example.

Example 2 *We have the L-Fuzzy context* (L, X, Y, R) *of the previous example and we take the label* $t = high$ *assigned to the values* $a = 0.8$ *and* $b = 1$. *The relation of the t-labeled L-Fuzzy context* (L, X, Y, R_{high}) *is in Table 2.*

Table 2. Relation R_{high}.

R_{high}	y_1	y_2	y_3	y_4
x_1	1	1	0.1	1
x_2	1	0.1	0	0
x_3	0.1	1	1	1
x_4	0	0.1	1	0.1
x_5	1	0.2	1	0

If we take $R(x_5, y_1) = 0.8$, *for instance, then we see that the new relation has the value* $R_t(x_5, y_1) = 1$. *In this case, the L-Fuzzy concepts associated with the sets created from* x_5 *and* y_1 *are:*

If we take $A = \{x_1/0, x_2/0, x_3/0, x_4/0, x_5/1\}$ *we obtain*

$$\{(x_1/0.1, x_2/0, x_3/0.1, x_4/0, x_5/1), (y_1/1, y_2/0.2, y_3/1, y_4/0)\}$$

and if we take $B = \{y_1/1, y_2/0, y_3/0, y_4/0\}$,

$$\{(x_1/1, x_2/1, x_3/0.1, x_4/0, x_5/1), (y_1/1, y_2/0.1, y_3/0, y_4/0)\}$$

As can be seen, Corollary 2 holds.

On the other hand, as the L-Fuzzy concepts obtained from R are:

From $A = \{x_1/0, x_2/0, x_3/0, x_4/0, x_5/1\}$:

$$\{(x_1/0.1, x_2/0, x_3/0.3, x_4/0.2, x_5/1), (y_1/0.8, y_2/0.2, y_3/1, y_4/0)\}$$

and from $B = \{y_1/1, y_2/0, y_3/0, y_4/0\}$:

$$\{(x_1/1, x_2/0.9, x_3/0.1, x_4/0, x_5/0.8), (y_1/1, y_2/0.2, y_3/0.1, y_4/0.1)\}$$

Then Corollary 1 also holds.

We will see now what happens with the more general departure sets (with several membership degrees equal to 1) and certain special labels.

Proposition 2 *Given an L-Fuzzy context (L, X, Y, R), a residuated implication operator I, and a label t, assigned to the values a and b such that $b = 1$, we can create the t-labeled L-Fuzzy context (L, X, Y, R_t) where $R_t(x, y) = R(x, y)_t$. Then the intension of the L-Fuzzy concept $(\bar{A}_{12}, \bar{A}_1)$ derived from any departure L-Fuzzy set $A \in L^X$ such that $A(x) = 0$ or 1, in the context (L, X, Y, R_t) is also equal to the intension of the L-Fuzzy concept (A_{12}, A_1) obtained in the L-Fuzzy context (L, X, Y, R) after applying label t. Analogous, we can write this proposition taking as a departure point a set $B \in L^Y$.*

Proof: Let (A_{12}, A_1) be the L-Fuzzy concept derived from $A \in L^X$ in the L-Fuzzy context (L, X, Y, R).

As the implication operator is residuated, $\forall z \in [0, 1]$, $I(0, z) = 1$ and $I(1, z) = z$ holds. Then as $A(x) = 0 \;\; 1$, it is true that:

$$\forall y \in Y, \quad \bar{A}_1(y) = \inf_{x \in X}\{I(A(x), R_t(x, y))\} = \inf_{x \in X/A(x)=1}(R_t(x, y))$$

and, as the label has a increasing compatibility function c_t :

$$A_1(y)_t = \left(\inf_{x \in X/A(x)=1} R(x, y)\right)_t = \inf_{x \in X/A(x)=1}(R_t(x, y)) = \bar{A}_1(y)$$

□

Example 3 *If we come back to the previous example where we used the label $t = high$ and we take the departure set $A = \{x_1/0, x_2/1, x_3/0, x_4/0, x_5/1\}$ in the t-labeled L-Fuzzy context (L, X, Y, R_{high}), then we obtain the L-Fuzzy concept:*

$$(\bar{A}_{12}, \bar{A}_1) = \{(x_1/1, x_2/1, x_3/0.1, x_4/0, x_5/1), (y_1/1, y_2/0.1, y_3/0, y_4/0)\}$$

On the other hand, if we have the initial L-Fuzzy context (L, X, Y, R) and the same departure set, then we obtain the L-Fuzzy concept:

$$(A_{12}, A_1) = \{(x_1/1, x_2/1, x_3/0.3, x_4/0.2, x_5/1), (y_1/0.8, y_2/0.1, y_3/0, y_4/0)\}$$

It is easy to see that the intension of the first one can be obtained applying label t to the intension of the second one.

Remark 1 *This result is not always true if we consider a label with a non increasing compatibility function or if the departure set has a membership degree different of 0 or 1, as can be seen in the following examples.*

Table 3. Relation $R_{medium-low}$.

$R_{medium-low}$	y_1	y_2	y_3	y_4
x_1	0	0	0.7	0
x_2	0.2	0.7	0.5	0.5
x_3	0.7	0	0.2	0.2
x_4	0.5	0.7	0	0.7
x_5	0.3	0.8	0	0.5

Example 4 *We have the L-Fuzzy context* (L, X, Y, R) *of the previous example and the label* $medium - low$ *assigned to the values* $a = 0.3$ *and* $b = 0.4$*. The relation* R *of the t-labeled L-Fuzzy context* $(L, X, Y, R_{medium-low})$ *is showed in Table 3.*

If we take $A = \{x_1/0, x_2/1, x_3/0, x_4/0, x_5/0\}$*, the intension of the L-Fuzzy concept associated with the initial L-Fuzzy context is* $A_1 = \{y_1/0.8, y_2/0.1, y_3/0, y_4/0\}$*, and the intension of the L-Fuzzy concept associated with the t-labeled L-Fuzzy context* $\bar{A}_1 = \{y_1/0.2, y_2/0.7, y_3/0, y_4/0.5\}$*, and, for instance,* $A_1(y_3)_{medium-low} \neq \bar{A}_1(y_3)$*, as can be seen.*

Example 5 *In the same L-Fuzzy context* (L, X, Y, R) *of the previous example, we take as a departure set* $A = \{x_1/0, x_2/0.5, x_3/0, x_4/0, x_5/1\}$*.*

The intension of the L-Fuzzy concept derived from A is $A_1 = \{y_1/0.8, y_2/0.2, y_3/0.6, y_4/0\}$*.*

If we consider the t-labeled L-Fuzzy context with $t = high$ *assigned to the values* $a = 0.8$ *and* $b = 1$*, represented in Table 2, then the intension of the L-Fuzzy concept obtained from A is* $\bar{A}_1 = \{y_1/1, y_2/0.2, y_3/0.5, y_4/0\}$*, and, as can be proved, we also have in this case* $A_1(y_3)_{high} \neq \bar{A}_1(y_3)$*.*

3.3 Using linguistic variables in departure sets

Once the previous properties have been studied, we return to the initial point that is to analyze how can we obtain the L-Fuzzy concepts associated with departure sets in which several labels take part. To do this, we will follow these steps:

1. Starting from the L-Fuzzy context (L, X, Y, R) and from the set of labels $T(V)$, we represent the situation that we want to study (departure set) by means of a set of pairs $\mathcal{P}_X = \{(x_i, t_{x_i}), x_i \in X, t_{x_i} \in T(V)\}$ which assigns labels of $T(V)$ to the elements of the set of objects X. In the same way, we define the set of pairs $\mathcal{P}_Y = \{(y_j, t_{y_j}), y_j \in Y, t_{y_j} \in T(V)\}$ for the set of attributes Y.
   Note that the same label could be associated with different objects or attributes.
2. We construct the t-labeled L-Fuzzy contexts (L, X, Y, R_t) associated with each of the labels used in the departure point as we defined in Definition 1.

3. For each pair (x_i, t_{x_i}) of $\mathcal{P}_X$ (or, analogously, for each pair (y_j, t_{y_j}) of $\mathcal{P}_Y$) we obtain the corresponding L-Fuzzy concept in the t-labeled L-Fuzzy context $(L, X, Y, R_{t_{x_i}})$. Taking as a departure the basic point:

$$A(x) = \begin{cases} 1 & \text{if } x = x_i \\ 0 & \text{otherwise} \end{cases}$$

4. We apply the intersection associated with the residuated implication operator that we are using to the intension (or extension) of the obtained L-Fuzzy concepts.

The obtained L-Fuzzy set is the one that we were looking for.

Let us see two examples:

Example 6 *Returning to the L-Fuzzy context (L, X, Y, R) of the previous example, we want now to analyze which objects verify that y_1 is high and y_3 is low (the values of the rest of the attributes do not matter to us). If we apply the exposed process:*

1. *In this case, we take the labels $\{high, low\}$ of the linguistic variable V choosing in high the values $a = 0.8$ and $b = 1$ and in low, $a = 0$ and $b = 0.2$ in the corresponding definition of the compatibility function.*
That is, our departure point will be represented by the pair $\mathcal{P}_Y = \{(y_1, high), (y_3, low)\}$.
2. *We consider now the t-labeled L-Fuzzy contexts (L, X, Y, R_{high}) and (L, X, Y, R_{low}) where the relations are the represented ones in Table 2 (used in the previous section) and Table 4.*

Table 4. R_{low} relation

R_{low}	y_1	y_2	y_3	y_4
x_1	0	0	1	0
x_2	0.1	1	1	1
x_3	1	0	0.1	0.1
x_4	1	1	0	1
x_5	0.2	1	0	1

3. *From the departure set $B = \{y_1/1, y_2/0, y_3/0, y_4/0\}$, we calculate the L-Fuzzy concept obtained in the t-labeled L-Fuzzy context (L, X, Y, R_{high}) to find the objects with high values of y_1:*

$$\{(x_1/1, x_2/1, x_3/0.1, x_4/0, x_5/1), (y_1/1, y_2/0.1, y_3/0, y_4/0)\}$$

And, we obtain the objects that have low values of y_3 calculating the L-Fuzzy concept in the t-labeled L-Fuzzy context (L, X, Y, R_{low}) from the set $B = \{y_1/0, y_2/0, y_3/1, y_4/0\}$:

$$\{(x_1/1, x_2/1, x_3/0.1, x_4/0, x_5/0), (y_1/0, y_2/0, y_3/1, y_4/0)\}$$

4. *Finally, we have to calculate the intersection of the extensions using the bounded difference* ($i(a,b) = \max(0, a+b-1)$) *associated with the implication of Lukasiewicz. The obtained result is* $\{(x_1/1, x_2/1, x_3/0, x_4/0, x_5/0)\}$, *and we can say that* x_1 *and* x_2 *are the objects that fulfill the initial condition.*

Example 7 *Returning to the first example, we wanted to see what objects shared attributes* y_2 *and* y_4 *but, in addition, did not have attributes* y_1 *and* y_3*.*
We apply the process:

1. *We also take the labels* $\{high, low\}$ *of the linguistic variable* V *choosing in high the values* $a = 0.8$ *and* $b = 1$ *and in low,* $a = 0$ *and* $b = 0.2$ *in the corresponding definition of the compatibility function. Our departure set is* $\mathcal{P}_Y = \{(y_1, low), (y_2, high), (y_3, low), (y_4, high)\}$.
2. *We consider now the t-labeled L-Fuzzy contexts* (L, X, Y, R_{high}) *and* (L, X, Y, R_{low}) *used in the previous example.*
3. *From the departure set* $B = \{y_1/0, y_2/1, y_3/0, y_4/1\}$, *we calculate the L-Fuzzy concept obtained in the t-labeled L-Fuzzy context* (L, X, Y, R_{high}) *to find the objects with high values of* y_2 *and* y_4*:*

$$\{(x_1/1, x_2/0, x_3/1, x_4/0.1, x_5/0), (y_1/0.1, y_2/1, y_3/0.1, y_4/1)\}$$

And, we obtain the objects that have low values of y_1 *and* y_3 *calculating the L-Fuzzy concept in the t-labeled L-Fuzzy context* (L, X, Y, R_{low}) *from the set* $B = \{y_1/1, y_2/0, y_3/1, y_4/0\}$*:*

$$\{(x_1/0, x_2/0.1, x_3/0.1, x_4/0, x_5/0), (y_1/1, y_2/0.9, y_3/1, y_4/1)\}$$

4. *Finally, the intersection of the extensions using the bounded difference associated with the implication of Lukasiewicz is* $\{(x_1/0, x_2/0, x_3/0.1, x_4/0, x_5/0)\}$, *and we can say that non object fulfills the initial condition, although if we had to choose one, then it would be object* x_3 *.*

4 Conclusions and future work

The use of linguistic variables in L-Fuzzy contexts is a good tool in knowledge acquisition processes since allow us represent our interest of study by means of an L-Fuzzy set and obtain the derived L-Fuzzy concept that give us the looked for information.

In future works we will study the use of these linguistic variables in the interval-valued L-Fuzzy contexts. In concrete, we will obtain significant relations, to replace erroneous values and to study interval-valued L-Fuzzy subcontexts.

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