Proto-fuzzy Concepts, their Retrieval and Usage

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Abstract. The aim of this paper is to define so-called proto-fuzzy concepts, as a base for generating different types of one-sided fuzzy concept lattices. Fuzzy formal context is a triple of a set of objects, a set of attributes and a fuzzy binary relation over a complete residuated lattice, which determines the degree of membership of each attribute to each object. A proto-fuzzy concept is a triple of a subset of objects, a subset of attributes and a value as the best common degree of membership of all pairs of objects and attributes from the above-mentioned sets to the fuzzy binary relation. Then the proto-fuzzy concepts will be found with a help of cuts and projections to the object-values or attribute-values plains of our fuzzy-context.

1 Introduction and motivation

Let us have a group of schoolmates of a secondary grammar school and their studying results of ten subjects as it is shown in the table below. Names of subjects are in the table as abbreviations (Ma – Math, Sl – Slovak language, Ph – Physics, Ge – Geography, Bi – Biology, Gr – German, En – English, Ch – Chemistry, Ae – Aesthetics, Hi – History). Abbreviations of names of students are in the table.

Table 1. Example of fuzzy formal context.

		Ma	Sl	Ph	Ge	Bi	Gr	En	Ch	Ae	Hi
F	Fred	1	1	1	3	2	1	2	2	1	2
J	Joey	3	1	2	1	1	1	1	3	1	1
A	Alice	3	2	3	1	1	1	1	3	2	2
N	Nancy	4	2	4	3	2	2	1	2	3	2
M	Mary	1	1	1	1	1	1	1	1	1	1
E	Eve	1	1	1	1	1	1	1	1	1	1
L	Lucy	1	3	1	2	2	2	2	1	2	2
D	David	2	3	4	3	4	1	1	2	2	2
P	Peter	2	1	2	1	1	2	2	3	1	2
T	Tom	1	3	2	2	2	2	2	3	1	2

The table is a concrete example of fuzzy formal context. Students represent objects, subjects represent attributes and corresponding valuations represent

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values assigned to every object–attribute pair by fuzzy binary relation over the set $\{1, 2, 3, 4, 5\}$ (1 – best, ..., 5 – worst). Goal is to find groups of students similar by their studying results of all subjects, or to find subsets of subjects similar by results of all students. In other words to find pairs of classical subset of objects or attributes and fuzzy subset of attributes or objects. Similarity is determined by fuzzy subsets. Those pairs are called one-sided fuzzy concepts ([1]).

The starting point of this paper is to define so-called *proto-fuzzy concepts*, triples made of a subset of objects, a subset of attributes and a value from the set of degrees of membership forming fuzzy binary relation, which is not exceeding for any object-attribute pair of cartesian product of object and attribute subsets meant above. Every element of the triple is "maximal" opposite to other two elements. Proto-fuzzy concepts can be taken as a "base structure unit" of one-sided fuzzy concepts. If values in the table are taken as columns tall as degree of membership of subsistent object–attribute pair to fuzzy binary relation, then proto-fuzzy concepts could be taken as a maximal "sub-blocks" of satisfying triples object-attribute-value of the 3D block representing fuzzy context. Examples of some proto-fuzzy concepts of the example will be shown in section 3.

2 Basic definitions

Definition 1. A formal context *is a triple* $\langle \mathcal{O}, \mathcal{A}, \mathcal{R} \rangle$ *consists of two sets* $\mathcal{O}$*, the set of* objects*, and* $\mathcal{A}$*, the set of* attributes*, and a relation* $\mathcal{R} \subseteq \mathcal{O} \times \mathcal{A}$.

Definition 2. A fuzzy formal context *is a triple* $\langle \mathcal{O}, \mathcal{A}, r \rangle$ *consists of two sets* $\mathcal{O}$*, the set of* objects*, and* $\mathcal{A}$*, the set of* attributes*, and* r *is fuzzy subset of* $\mathcal{O} \times \mathcal{A}$*, mapping from* $\mathcal{O} \times \mathcal{A}$ *to* L*, where* L *is a lattice.*

In the sense of simplicity of an idea "fuzzy" will be used instead of L-fuzzy.

Definition 3. *For every* $l \in L$ *define mappings* $\uparrow_l$: $\mathcal{P}(\mathcal{O}) \to \mathcal{P}(\mathcal{A})$ *and* $\downarrow_l$: $\mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{O})$*: For every subset* $O \subseteq \mathcal{O}$ *put*

$$\uparrow_l (O) = \{a \in \mathcal{A} : (\forall o \in O) r(o, a) \geq l\}$$

and for all $A \subseteq \mathcal{A}$ *put*

$$\downarrow_l (A) = \{o \in \mathcal{O} : (\forall a \in A) r(o, a) \geq l\}.$$

Lemma 1. *For all* $l \in L$ *the pair* $(\uparrow_l, \downarrow_l)$ *forms a Galois connection between the power-set lattices* $\mathcal{P}(\mathcal{O})$ *and* $\mathcal{P}(\mathcal{A})$.

Definition 4. *Let* $\langle \mathcal{O}, \mathcal{A}, r \rangle$ *be a fuzzy context. A pair* $\langle O, A \rangle$ *is called an* l-concept *iff* $\uparrow_l (O) = A$*, and* $\downarrow_l (A) = O$*, hence the pair is a concept in a classical context* $\langle \mathcal{O}, \mathcal{A}, \mathcal{R}_l \rangle$*, that*

$$\mathcal{R}_l = \{(o, a) \in \mathcal{O} \times \mathcal{A} : r(o, a) \geq l\}.$$

Context $\langle \mathcal{O}, \mathcal{A}, \mathcal{R}_l \rangle$ *is called an* l-cut*. The set of all concepts in an* l*-cut will be denoted* $\mathcal{K}_l$.

Table 2. 2-cut.

2	Ma	Sl	Ph	Ge	Bi	Gr	En	Ch	Ae	Hi
F	•	•	•		•	•	•	•	•	•
J		•	•	•	•	•	•		•	•
A		•		•	•	•	•		•	•
N		•			•	•	•	•		•
E	•	•	•	•	•	•	•	•	•	•
M	•	•	•	•	•	•	•	•	•	•
L	•		•	•	•	•	•	•	•	•
D	•					•	•	•	•	•
P	•	•	•	•	•	•	•		•	•
T	•		•	•	•	•	•		•	•

In our example the l-cut means a look at the level of success for the value l. So the l-cut gives an Yes/No answer for the question: *Is the result of each student in each subject at least of the value l?* For example a concept $\langle O, A\rangle$ from $\mathcal{K}_2$ represents the group O of students, that every subject of the set A is fulfilled at least in the value 2.

By exploring all l-cuts for such $l \in L$, it can be seen that some l-concepts are equal for different $l \in L$. But information that Eve and Mary are successful in all subjects for the value 2 is not complete and not as useful as information that they are successful for 1. This information is not complete, "closed".

Two interesting properties will be shown in following lemmas and theorems. It will be a continuation of the knowledge of the paper [3], where some properties of cuts was shown.

Lemma 2. *Let $l_1, l_2 \in L$ that $l_1 \leq l_2$. $\uparrow_{l_1} (O) \supseteq \uparrow_{l_2} (O)$ for every $O \subseteq \mathcal{O}$ and $\downarrow_{l_1} (A) \supseteq \downarrow_{l_2} (A)$ for every $A \subseteq \mathcal{A}$.*

Proof. The proof will be shown for $\uparrow$. The proof for $\downarrow$ is likewise.

If $l_1 \leq l_2$ then

$$\{a \in \mathcal{A} : (\forall o \in O) r(o, a) \geq l_1\} \supseteq \{a \in \mathcal{A} : (\forall o \in O) r(o, a) \geq l_2\}.$$

Hence $\uparrow_{l_1} (O) \supseteq \uparrow_{l_2} (O)$. □

Lemma 3. *Let $O \subseteq \mathcal{O}$, $A \subseteq \mathcal{A}$ and $l_1, l_2 \in L$. Then $\uparrow_{l_1} (O) \cap \uparrow_{l_2} (O) = \uparrow_{l_1 \vee l_2} (O)$ and $\downarrow_{l_1} (A) \cap \downarrow_{l_2} (A) = \downarrow_{l_1 \vee l_2} (A)$.*

Proof. If $a \in \uparrow_{l_1} (O) \cap \uparrow_{l_2} (O)$ then for all $o \in O$ is $r(o, a) \geq l_1$ and $r(o, a) \geq l_1$. It follows from above that for every $o \in O$ is $r(o, a) \geq l_1 \vee l_2$ and so $a \in \uparrow_{l_1 \vee l_2} (O)$. Hence $\uparrow_{l_1} (O) \cap \uparrow_{l_2} (O) \subseteq \uparrow_{l_1 \vee l_2} (O)$. The lemma 6 implies that $\uparrow_{l_1 \vee l_2} (O) \subseteq \uparrow_{l_1} (O)$ and $\uparrow_{l_1 \vee l_2} (O) \subseteq \uparrow_{l_2} (O)$. It implies that $\uparrow_{l_1 \vee l_2} (O) \subseteq \uparrow_{l_1} (O) \cap \uparrow_{l_2} (O)$. From the both inclusions implies that $\uparrow_{l_1 \vee l_2} (O) = \uparrow_{l_1} (O) \cap \uparrow_{l_2} (O)$. The proof for $\downarrow$ is likewise. □

Theorem 1. *Let $l_1, l_2 \in L$ and $\langle O, A\rangle \in \mathcal{K}_{l_1} \cap \mathcal{K}_{l_2}$. Then for all $l \in L$, if $l_1 \leq l \leq l_2$ then $\langle O, A\rangle \in \mathcal{K}_l$.*

Proof. The lemma 6 and $\langle O, A\rangle \in \mathcal{K}_{l_1} \cap \mathcal{K}_{l_2}$ implies that
$A =\uparrow_{l_1} (O) \supseteq\uparrow_l (O) \supseteq\uparrow_{l_2} (O) = A$,
$O =\downarrow_{l_1} (A) \supseteq\downarrow_l (A) \supseteq\downarrow_{l_2} (A) = O$
Hence $\uparrow_l (O) = A$ and $\downarrow_l (A) = O$, which implies $\langle O, A\rangle \in \mathcal{K}_l$. □

Theorem 2. *Let $l_1, l_2 \in L$ and $\langle O, A\rangle \in \mathcal{K}_{l_1} \cap \mathcal{K}_{l_2}$. Then $\langle O, A\rangle \in \mathcal{K}_{l_1 \vee l_2}$.*

Proof. The lemma 7 implies
$\uparrow_{l_1\vee l_2} (O) =\uparrow_{l_1} (O) \cap \uparrow_{l_2} (O) = A \cap A = A$,,
$\downarrow_{l_1\vee l_2} (A) =\downarrow_{l_1} (A) \cap \downarrow_{l_2} (A) = O \cap O = O$.
Hence $\langle O, A\rangle \in \mathcal{K}_{l_1\vee l_2}$. □

3 Proto-fuzzy concepts and their usage

Definition 5. *Triples $\langle O, A, l\rangle \in \mathcal{P}(\mathcal{O}) \times \mathcal{P}(\mathcal{A}) \times L$ such that $\langle O, A\rangle \in \bigcup_{k\in L} \mathcal{K}_k$ and $l = \sup\{k \in L : \langle O, A\rangle \in \mathcal{K}_k\}$ will be called* proto-fuzzy concepts. *The set of all proto-fuzzy concepts will be denoted $\mathcal{K}^P$.*

For our example will proto-fuzzy concept $\langle O, A, l\rangle$ means the group of students O, whose best common result of all subjects from the set A is l. In the following tables are some proto-fuzzy concepts of our example.

{F, J, A, N, M, E, L, D, P, T}	{F, J, A, P, E, M}	{F, M, E, L}
{Sl, Ge, Gr, En, Ch, Ae, Hi}	{Sl, Bi, Ae , Gr, En , Hi}	{Ma, Ph}
3	2	1

The set of all proto-fuzzy concepts will be used for creating one-sided fuzzy concepts with help of mappings defined below. Mappings will determine which side will be fuzzy.

Definition 6. *Let $O \subseteq \mathcal{O}$ be an arbitrary set of objects. The set*

$$\mathcal{K}^P_O = \{\langle A, l\rangle \in \mathcal{P}(\mathcal{A}) \times L : (\exists B \supseteq O)\langle B, A, l\rangle \in \mathcal{K}^P\}$$

will be called the contraction *of the set of proto-fuzzy concepts subsistent to the set O.*

Definition 7. *Define mappings*

$$\Uparrow : 2^{\mathcal{O}} \to L^{\mathcal{A}},$$

$$\Downarrow : L^{\mathcal{A}} \to 2^{\mathcal{O}}$$

in the following way: For every subset O of objects and for every fuzzy-subsets of attributes put

$$\Uparrow (O)(a) = \sup\{l \in L : (\exists\langle A, l\rangle \in \mathcal{K}^P_O) a \in A\}$$

$$\Downarrow (\widetilde{A}) = \bigcup\{O \subseteq \mathcal{O} : (\forall a \in \mathcal{A})(\exists\langle A, l\rangle \in \mathcal{K}^P_O) a \in A \ \&\ l \geq \widetilde{A}(a)\}.$$

Lemma 4. *Let O and A are arbitrary subsets of objects and attributes respectively, and l be an arbitrary value of L such that for every object o of the set O and for every attribute a of the set A, $\mathcal{R}(o,a) \geq l$. Then there exist subsets $\overline{O} \supseteq O$, $\overline{A} \supseteq A$ and value $k \in L$ such that $k \geq l$ and $\langle \overline{O}, \overline{A}, k \rangle \in \mathcal{K}^P$.*

Proof. It is given that $(\forall o \in O)(\forall a \in A) r(o,a) \geq l$. Take

$$\overline{A} = \uparrow_l (O) = \{a \in \mathcal{A} : (\forall o \in O) r(o,a) \geq l\} \supseteq A.$$

Then

$$\overline{O} = \downarrow_l (\overline{A}) = \downarrow_l (\uparrow_l (O))$$

and from the fact that for every $l \in L$ the pair $(\uparrow_l, \downarrow_l)$ forms a Galois connection, it implies that $\downarrow_l (\uparrow_l (O)) \supseteq O$ and hence $\langle \overline{O}, \overline{A} \rangle \in \mathcal{K}_l$. If

$$k = \sup\{m \in L : \langle \overline{O}, \overline{A} \rangle \in \mathcal{K}_m\}$$

the theorem 9 implies that $\langle \overline{O}, \overline{A} \rangle \in \mathcal{K}_k$ and so

$$\langle \overline{O}, \overline{A}, k \rangle \in \mathcal{K}^P.$$

□

Lemma 5. *Let $l \in L$, $O_1, O_2 \subseteq \mathcal{O}$ and $\langle A_1, l_1 \rangle \in \mathcal{K}^P_{O_1}$, $\langle A_2, l_2 \rangle \in \mathcal{K}^P_{O_2}$ that $A_1 \cap A_2 \neq \emptyset$ and $l_1 \wedge l_2 \geq l$. Then exists $\langle A, k \rangle \in \mathcal{K}^P_{O_1 \cup O_2}$ that $A \supseteq A_1 \cap A_2$ and $k \geq l$.*

Proof. $\langle A_1, l_1 \rangle \in \mathcal{K}^P_{O_1}$ it means that

$$(\forall o \in O_1)(\forall a \in A_1) r(o,a) \geq l_1.$$

$\langle A_2, l_2 \rangle \in \mathcal{K}^P_{O_2}$ it means that

$$(\forall o \in O_2)(\forall a \in A_2) r(o,a) \geq l_2.$$

Hence

$$(\forall o \in O_1 \cup O_2)(\forall a \in A_1 \cap A_2) r(o,a) \geq l_1 \wedge l_2 \geq l.$$

The lemma 13 implies that

$$(\exists O \supseteq O_1 \cup O_2)(\exists A \supseteq A_1 \cap A_2)(\exists k \in L : k \geq l) \langle O, A, k \rangle \in \mathcal{K}^P$$

hence

$$\langle A, k \rangle \in \mathcal{K}^P_{O_1 \cup O_2}.$$

□

Lemma 6. *Let $O \subseteq \mathcal{O}$, $\langle A_1, l_1 \rangle$, $\langle A_2, l_2 \rangle \in \mathcal{K}^P_O$ such that $A_1 \cap A_2 \neq \emptyset$. Then there exist $A \subseteq \mathcal{A}$ and $l \geq l_1 \vee l_2$ such that $\langle A, l \rangle \in \mathcal{K}^P_O$.*

Proof. For all $o \in O$ and for all $a \in A_1 \cap A_2$ is

$$r(o,a) \geq l_1 \text{ and } r(o,a) \geq l_2.$$

Hence

$$r(o,a) \geq l_1 \vee l_2.$$

From above and lemma 13 implies that there exist

$$(\exists B \supseteq O)(\exists A \supseteq A_1 \cap A_2)(\exists l \in L : l \geq l_1 \vee l_2)\langle B, A, k\rangle \in \mathcal{K}^P.$$

Hence

$$\langle A, l\rangle \in \mathcal{K}^P_O.$$

□

Lemma 7. *Let O_1, O_2 be an arbitrary subsets of the set of objects such that $O_1 \subseteq O_2$. Then $\mathcal{K}^P_{O_1} \supseteq \mathcal{K}^P_{O_2}$.*

Proof. Because of $O_1 \subseteq O_2$ is

$$\{\langle A_1, l_1\rangle \in \mathcal{P}(A) \times L : (\exists B_1 \supseteq O_1)\langle B_1, A_1, l_1\rangle \in \mathcal{K}^P\} \supseteq$$

$$\supseteq \{\langle A_2, l_2\rangle \in \mathcal{P}(A) \times L : (\exists B_2 \supseteq O_2)\langle B_2, A_2, l_2\rangle \in \mathcal{K}^P\}.$$

Hence

$$\mathcal{K}^P_{O_1} \supseteq \mathcal{K}^P_{O_2}.$$

□

Theorem 3. *The pair of mappings $(\Uparrow, \Downarrow)$ forms a Galois connection between the power-set lattice $\mathcal{P}(\mathcal{O})$ and the fuzzy power-set lattice $\mathcal{F}(\mathcal{A})$.*

Proof. For every set O, the subset of the set of objects and the fuzzy set $\widetilde{A}$, the fuzzy-subset of the set of attributes, have to be proven that O is the subset of $\Downarrow(\widetilde{A})$ if, and only if $\widetilde{A}$ is the fuzzy-subset of $\Uparrow(O)$.
$\boxed{\Rightarrow}$

$$O \subseteq \Downarrow(\widetilde{A}) = \bigcup\{B \subseteq \mathcal{O} : (\forall b \in \mathcal{A})(\exists \langle A, l\rangle \in \mathcal{K}^P_B) b \in A \;\&\; l \geq \widetilde{A}(b)\}.$$

Let $a \in \mathcal{A}$ be an arbitrary attribute. The lemma 14 implies that there exists $A_a \subseteq \mathcal{A}$, $l_a \in L$ such that $a \in A_a$, $l_a \geq \widetilde{A}(a)$ and

$$\langle A_a, l_a\rangle \in \mathcal{K}^P_{\Downarrow(\widetilde{A})}.$$

$O \subseteq \Downarrow(\widetilde{A})$ implies that $\mathcal{K}^P_O \supseteq \mathcal{K}^P_{\Downarrow(\widetilde{A})}$. Hence $\langle A_a, l_a\rangle \in \mathcal{K}^P_O$. So

$$\widetilde{A}(a) \leq l_a \leq \sup\{l \in L : (\exists \langle A, l\rangle \in \mathcal{K}^P_O) a \in A\} = \Uparrow(O)(a).$$

Because of arbitrarity of attribute a and from unequality above implies that $\widetilde{A}$ is the fuzzy-subset of $\Uparrow(O)$.
$\boxed{\Leftarrow}$ Let $a \in \mathcal{A}$ be an arbitrary attribute. Denote

$$l_a = \Uparrow(O)(a) = \sup\{l \in L : (\exists \langle A, l\rangle \in \mathcal{K}_O^P) a \in A\}.$$

The proposition implies that for every $a \in \mathcal{A}$, $\widetilde{A}(a) \leq l_a$. The lemma 15 implies that there exists $A_a \subseteq \mathcal{A}$ such that $\langle A_a, l_a\rangle \in \mathcal{K}_O^P$, and that implies

$$O \in \{B \subseteq \mathcal{O} : (\forall b \in \mathcal{A})(\exists\langle A, l\rangle \in \mathcal{K}_B^P) a \in A \ \&\ l \geq \widetilde{A}(b)\}$$

hence

$$O \subseteq \bigcup\{B \subseteq \mathcal{O} : (\forall b \in \mathcal{A})(\exists\langle A, l\rangle \in \mathcal{K}_B^P) a \in A \ \&\ l \geq \widetilde{A}(b)\} = \Downarrow(\widetilde{A}).$$

So the set O is subset of $\Downarrow(\widetilde{A})$. □

For the case of object fuzzy side will be used mappings:

$$\Uparrow : 2^{\mathcal{A}} \to L^{\mathcal{O}},$$

$$\Downarrow : L^{\mathcal{O}} \to 2^{\mathcal{A}}.$$

Let $\widetilde{O}$ be a fuzzy subset of objects and $A \subseteq \mathcal{A}$ is subset of attributes.

$$\Uparrow(A)(o) = \sup\{l \in L : (\exists\langle O, l\rangle \in \mathcal{K}_A^P) o \in O\}$$

$$\Downarrow(\widetilde{O}) = \bigcup\{T \subseteq \mathcal{A} : (\forall o \in \mathcal{O})(\exists\langle O, l\rangle \in \mathcal{K}_T^P) o \in O \ \&\ l \geq \widetilde{O}(a)\},$$

where

$$\mathcal{K}_A^P = \{\langle O, l\rangle : (\exists T \supseteq A)\langle O, T, l\rangle \in \mathcal{K}^P\}.$$

Example 1. For example take the fuzzy-subset of the set of attributes,

$$\widetilde{A} = \{(\text{Ma},1), (\text{Sl},3), (\text{Ph},1), (\text{Ge},3), (\text{Bi},4), (\text{Gr},2), (\text{En},2), (\text{Ch},2), (\text{Ae},4), (\text{Hi},4)\}.$$

In the table below are some proto-fuzzy concepts which contains students whose results satisfy to $\widetilde{A}$. Hence $\Downarrow(\widetilde{A})$ ={F,L,M,E}. Elements of $K^P_{\Downarrow(\widetilde{A})}$ are shown in the next table. Hence

$$\Uparrow(\Downarrow(\widetilde{A})) =$$

$$= \{(\text{Ma},1),(\text{Sj},3),(\text{Ph},1),(\text{Ge},3),(\text{Bi},2),(\text{Gr},2),(\text{En},2),(\text{Ch},2),(\text{Ae},2),(\text{Hi},2)\}$$

.

Table 3. Some of proto-fuzzy concepts which satisfy to $\widetilde{A}$

{M, E}	{Ma,Sl,Ph,Ge,Bi,Gr,En,Ch,Ae,Hi}	1
{M, E, F}	{Ma, Sl, Ph, Gr, Ae, Hi}	1
{M, E, L}	{Ma, Ph, Ch}	1
{M, E, F, L}	{Ma, Ph}	1
{M, E, F}	{Ma,Sl,Ph,Bi,Ch,Ae,En,Gr,Hi}	2
{M, E, L}	{Ma,Ph,Ge,Bi,Ch,Ae,En,Gr,Hi}	2
{M, E, F, L}	{Ma,Ph,Bi,Ch,Ae,En,Gr,Hi}	2

Table 4. Elements of the $K^P_{\Downarrow(\widetilde{A})} = K^P_{\{\mathrm{M,E,F,L}\}}$

{Ma,Sl,Ph,Ge,Bi,Gr,En,Ch,Ae,Hi}	3
{Ma, Ch, Ae, Gr, En, Hi}	2
{Ma, Ph, Bi, Ch, Ae, Gr, En, Hi}	2
{Ma,Ph,Bi,Gr,En,Ae,Hi}	2
{Ae,Gr,En,Hi}	2
{Ph,Bi,Gr,En,Ae,Hi}	2
{Bi,Gr,En,Ae,Hi}	2
{Bi,Gr,En,Ae,Hi}	2
{Ch,Gr,En,Hi}	2
{Ma,Gr,En,Ae,Hi}	2
{Ma,Ph}	1

4 Retrieval of proto-fuzzy concepts

Proto-fuzzy concepts will be retrieved with a help of cuts and "pessimistic sights" to object-value or attribute-value plains.

Definition 8. *Define new binary relations*

$$\mathcal{R}_{\mathcal{A}} = \{(o, l) \in \mathcal{O} \times L : (\forall a \in \mathcal{A}) r(o, a) \geq l\}$$

and

$$\mathcal{R}_{\mathcal{O}} = \{(a, l) \in \mathcal{A} \times L : (\forall o \in \mathcal{O}) r(o, a) \geq l\}.$$

The formal context $\langle \mathcal{O}, L, \mathcal{R}_{\mathcal{A}} \rangle$ *will be called* object–value sight *and the formal context* $\langle \mathcal{A}, L, \mathcal{R}_{\mathcal{O}} \rangle$ *will be called* attribute–value sight.

Table 5. Object–value and attribute–value sight

	1	2	3	4	5
Fred			•	•	•
Joey			•	•	•
Alice			•	•	•
Nancy				•	•
Mary	•	•	•	•	•
Eve	•	•	•	•	•
Lucy			•	•	•
David				•	•
Peter			•	•	•
Tom			•	•	•

	1	2	3	4	5
Math				•	•
Slovak language			•	•	•
Physics				•	•
Geography			•	•	•
Biology				•	•
German language		•	•	•	•
English language		•	•	•	•
Chemistry			•	•	•
Aesthetics			•	•	•
History		•	•	•	•

Definition 9. *Define new mappings*

$$\uparrow_{\mathcal{A}} : 2^{\mathcal{O}} \to L \text{ and } \downarrow_{\mathcal{A}} : L \to 2^{\mathcal{O}},$$

$$\uparrow_{\mathcal{O}} : 2^{\mathcal{A}} \to L \text{ and } \downarrow_{\mathcal{O}} : L \to 2^{\mathcal{A}}.$$

For every $O \subseteq \mathcal{O}$, $A \subseteq \mathcal{A}$ *and* $l \in L$ *put*

$$\uparrow_{\mathcal{A}} (O) = \inf\{\sup\{l \in L : (o, l) \in \mathcal{R}_{\mathcal{A}}\} : o \in O\}$$

$$\downarrow_{\mathcal{A}} (l) = \{o \in \mathcal{O} : (o, l) \in \mathcal{R}_{\mathcal{A}}\}$$

$$\uparrow_{\mathcal{O}} (A) = \inf\{\sup\{l \in L : (a, l) \in \mathcal{R}_{\mathcal{O}}\} : a \in A\}$$

$$\downarrow_{\mathcal{O}} (l) = \{a \in \mathcal{A} : (a, l) \in \mathcal{R}_{\mathcal{O}}\}.$$

Theorem 4. *Pairs of mappings $(\uparrow_{\mathcal{A}}, \downarrow_{\mathcal{A}})$ and $(\uparrow_{\mathcal{O}}, \downarrow_{\mathcal{O}})$ form Galois connections between the power-set lattice $\mathcal{P}(O)$ or $\mathcal{P}(A)$ and the lattice of values L.*

Proof. The proof will be shown only for first pair. The proof for second pair is likewise.

1. Let $O_1 \subseteq O_2 \subseteq \mathcal{O}$. It follows from an inclusion above that

$$\{\sup\{l \in L : (o,l) \in \mathcal{R}_{\mathcal{A}}\} : o \in O_1\} \subseteq$$

$$\subseteq \{\sup\{l \in L : (o,l) \in \mathcal{R}_{\mathcal{A}}\} : o \in O_2\}$$

and from a properties of infimum

$$\inf\{\sup\{l \in L : (o,l) \in \mathcal{R}_{\mathcal{A}}\} : o \in O_1\} \geq$$

$$\geq \inf\{\sup\{l \in L : (o,l) \in \mathcal{R}_{\mathcal{A}}\} : o \in O_2\}.$$

Hence

$$\uparrow_{\mathcal{A}}(O_1) \geq \uparrow_{\mathcal{A}}(O_2).$$

2. Let $l_1, l_2 \in L$. If $l_1 \leq l_2$ then

$$\{o \in \mathcal{O} : (o, l_1) \in \mathcal{R}_{\mathcal{A}}\} \supseteq \{o \in \mathcal{O} : (o, l_2) \in \mathcal{R}_{\mathcal{A}}\}.$$

Hence

$$\downarrow_{\mathcal{A}}(l_1) \supseteq \downarrow_{\mathcal{A}}(l_2).$$

3. Let $O \subseteq \mathcal{O}$. Denote

$$s_o = \sup\{l \in L : (o,l) \in \mathcal{R}_{\mathcal{A}}\},$$

for arbitrary object $o \in O$. From definition of $\uparrow_{\mathcal{A}}$

$$\uparrow_{\mathcal{A}}(O) = \inf\{s_b : b \in O\} \leq s_o$$

and from property 2 implies

$$\downarrow_{\mathcal{A}}(\uparrow_{\mathcal{A}}(O)) \supseteq \downarrow_{\mathcal{A}}(s_o) = \{b \in \mathcal{O} : (b, s_o) \in \mathcal{R}_{\mathcal{A}}\}.$$

Arbitrarity of o implies that

$$\downarrow_{\mathcal{A}}(\uparrow_{\mathcal{A}}(O)) \supseteq \bigcup_{o \in O}\{b \in \mathcal{O} : (b, s_o) \in \mathcal{R}_{\mathcal{A}}\} \supseteq O.$$

4. Let $l \in L$ be an arbitrary value. Denote $s_o = \sup\{k \in L : (o,k) \in \mathcal{R}_{\mathcal{A}}\}$. For all

$$o \in \downarrow_{\mathcal{A}}(l) = \{b \in \mathcal{O} : (b,l) \in \mathcal{R}_{\mathcal{A}}\}$$

is $s_o \geq l$. Hence

$$\uparrow_{\mathcal{A}}(\downarrow_{\mathcal{A}}(l)) = \inf\{s_b : b \in \downarrow_{\mathcal{A}}(l)\} \geq l.$$

□

Definition 10. *The pair* $\langle O, l\rangle$ *is called* $\mathcal{A}$-concept *of the object–value sight* $\langle \mathcal{O}, L, \mathcal{R}_{\mathcal{A}}\rangle$ *iff* $\uparrow_{\mathcal{A}} (O) = l$ *and* $\downarrow_{\mathcal{A}} (l) = O$. *The set of all* $\mathcal{A}$*-concepts will be denoted* $\mathcal{K}_{\mathcal{A}}$.

Definition 11. *The pair* $\langle A, l\rangle$ *is called* $\mathcal{O}$-concept *of the attribute–value sight* $\langle \mathcal{A}, L, \mathcal{R}_{\mathcal{O}}\rangle$ *iff* $\uparrow_{\mathcal{O}} (A) = l$ *and* $\downarrow_{\mathcal{O}} (l) = A$. *The set of all* $\mathcal{O}$*-concepts will be denoted* $\mathcal{K}_{\mathcal{O}}$.

It can be defined an object–value sight for every subset of attributes or attribute–value sight for every subset of objects, but their usage for this paper wasn't necessary.

Theorem 5. *Let* $l \in L$, $A_1, A_2 \subseteq \mathcal{A}$, $O_1, O_2 \subseteq \mathcal{O}$ *such that* $\langle \mathcal{O}, A_1, l\rangle, \langle O_1, \mathcal{A}, l\rangle \in \mathcal{K}^P$ *and* $\langle O_2, A_2\rangle \in \mathcal{K}_l$ *for context* $\langle \mathcal{O} \setminus O_1, \mathcal{A} \setminus A_1, \mathcal{R}_l\rangle$. *Then*

$$\langle O_1 \cup O_2, A_1 \cup A_2, l\rangle, \in \mathcal{K}^P.$$

Proof. It will be shown that $A_1 \cup A_2 =\downarrow_l (O_1 \cup O_2)$ and $O_1 \cup O_2 =\uparrow_l (A_1 \cup A_2)$.

If $a \in A_1$ then for all $o \in \mathcal{O}$ is $(o, a) \in \mathcal{R}_l$.

If $a \in A_2$ then for all $o \in O_1 \cup O_2$ is $(o, a) \in \mathcal{R}_l$.

If $a \in A_1 \cup A_2$ then for all $o \in (\mathcal{O} \cap (O_1 \cup O_2)) = O_1 \cup O_2$ is $(o, a) \in \mathcal{R}_l$.

Hence $A_1 \cup A_2 \subseteq\uparrow_l (O_1 \cup O_2)$.

The opposite inclusion will be shown by contradiction. Let us assume $a \in\uparrow_l (O_1 \cup O_2)$ and $a \notin A_1 \cup A_2$.

From $a \in\uparrow_l (O_1 \cup O_2)$ implies that for all $o \in O_1 \cup O_2 \supseteq O_2$ is $(o, a) \in \mathcal{R}_l$. From $a \notin A_1 \cup A_1$ implies that $a \in (\mathcal{A} \setminus (A_1 \cup A_1)) = ((\mathcal{A} \setminus A_1) \setminus A_2)$. It is the contradiction to precondition $\langle O_2, A_2\rangle \in \mathcal{K}_l$ for context $\langle \mathcal{O} \setminus O_1, \mathcal{A} \setminus A_1, \mathcal{R}_l\rangle$.

The second equality can be shown likewise. $\square$

Subcontexts from the theorem will be called *auxiliary subcontexts of l-cut*. Concepts of sights will be retrieved with a help of mappings $\uparrow_{\mathcal{A}}, \downarrow_{\mathcal{A}}, \uparrow_{\mathcal{O}}$ and $\downarrow_{\mathcal{O}}$. It's good to know that $\langle O, l\rangle \in \mathcal{K}_{\mathcal{A}}$ then $\langle O, \mathcal{A}, l\rangle \in \mathcal{K}^P$, because of $\mathcal{A}$ is closed. Denote $\mathcal{A}$ as the set of all subjects and $\mathcal{O}$ as the group of all students from our example. Hence

$$\langle \mathcal{O}, \mathcal{A}, 4\rangle \in \mathcal{K}^P,$$

$$\langle \mathcal{O} \setminus \{\text{N,D}\}, \mathcal{A}, 3\rangle, \langle \mathcal{O}, \mathcal{A} \setminus \{\text{Ma,Ph,Bi}\}, 3\rangle \in \mathcal{K}^P,$$

$$\langle \{\text{M,E}\}, \mathcal{A}, 1\rangle, \langle \mathcal{O}, \{\text{Gr,En,Hi}\}, 2\rangle \in \mathcal{K}^P,$$

Let us create auxiliary subcontexts of 3-cut, 2-cut and 1-cut.

Table 6. Auxiliary subcontexts of 3-cut

3	Ma	Ph	Bi
N			•
D	•		

There are only two concepts in the auxiliary subcontext of 3-cut, $\langle\{N\},\{Bi\}\rangle$ and $\langle\{D\},\{Ma\}\rangle$. The theorem 23 implies that
$\langle\mathcal{O}\setminus\{\mathrm{N}\},\mathcal{A}\setminus\{Ph,Bi\},3\rangle$, $\langle\mathcal{O}\setminus\{D\},\mathcal{A}\setminus\{\mathrm{Ma,Ph}\},3\rangle\in\mathcal{K}^P$.

Table 7. Auxiliary subcontext of 2-cut

2	Ma	Sl	Ph	Ge	Bi	Ch	Ae
F	•	•	•		•	•	•
J		•	•	•	•		•
A		•		•	•		•
N		•			•	•	
L	•		•	•	•	•	•
D	•					•	•
P	•	•	•	•	•		•
T	•		•	•	•		•

Because of the convexity of l-concepts, we can omit Eve and Mary from the set of students for auxiliary subcontext of 2-cut. And for input of theorem 23 for degree 2 can be used proto-fuzzy concepts $\langle\{\mathrm{M,E}\},\mathcal{A},1\rangle,\langle\mathcal{O},\{\mathrm{Gr,En,Hi}\},2\rangle\in\mathcal{K}^P$.

Table 8. Auxiliary subcontext of 1-cut

1	Ma	Sl	Ph	Ge	Bi	Gr	En	Ch	Ae	Hi
F	•	•	•			•			•	
J		•		•	•	•	•		•	•
A				•	•	•	•			
N							•			
L	•		•					•		
D						•	•			
P		•		•	•				•	
T	•								•	

5 Conclusion

Conceptual scaling and theory of triadic contexts will be the object of our future work and study. We will try to algoritmize outline process.

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