On Generalization of Fuzzy Concept Lattices Based on Change of Underlying Fuzzy Order

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Abstract. The paper presents a generalization of the main theorem of fuzzy concept lattices. The theorem is investigated from the point of view of fuzzy logic. There are various fuzzy order types which differ by incorporated relation of antisymmetry. This paper focuses on fuzzy order which uses fuzzy antisymmetry defined by means of multiplication operation and fuzzy equality.

Keywords. Fuzzy order, fuzzy concept lattice

1 Introduction

A notion of fuzzy order has been derived from the classical one by fuzzification the three underlying relations. This led to various versions, at the beginning versions utilizing the classical relation of equality (see e.g. [9]); later versions are more general by introducing fuzzy similarity (or fuzzy equality) instead of the classical equality (see e.g. [4]). Fuzzy similarity is based on idea that relationship between objects A and B should be transformed to a similar relationship between objects A' and B' whenever A', B' are similar to A, B, respectively.

In [3], one of the later definitions of fuzzy order was used to formulate and prove a fuzzy logic extension of the main theorem of concept lattices. The aim of this paper is to enlarge validity of the theorem to more general fuzzy order.

2 Preliminaries

First, we recall some basic notions. It is known that in fuzzy logic an important structure of truth values is represented by a complete residuated lattice (see e.g. [5], [6], [7]).

Definition 1. *A residuated lattice is an algebra* $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ *such that*

- $\langle L, \wedge, \vee, 0, 1\rangle$ *is a lattice with the least element* 0 *and the greatest element* 1*,*
- $\langle L, \otimes, 1\rangle$ *is a commutative monoid,*

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– *$\otimes$ and $\rightarrow$ form so-called adjoint pair, i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ holds for all $a, b, c \in L$.*

Residuated lattice **L** *is called complete if $\langle L, \wedge, \vee \rangle$ is a complete lattice.*

Throughout the paper, **L** will denote a complete residuated lattice. An **L**-*set* (or fuzzy set) A in a universe set X is any mapping $A : X \rightarrow L$, $A(x)$ being interpreted as the truth degree of the fact that "x belongs to A". By L^X we denote the set of all **L**-sets in X. A binary **L**-relation is defined obviously. Operations on L extend pointwise to L^X, e.g. $(A \vee B)(x) = A(x) \vee B(x)$ for any $A, B \in L^X$. As is usual, we write $A \cup B$ instead of $A \vee B$, etc.

L-*equality* (or fuzzy equality) is a binary **L**-relation $\approx \in L^{X \times X}$ such that $(x \approx x) = 1$ (reflexivity), $(x \approx y) = (y \approx x)$ (symmetry), $(x \approx y) \otimes (y \approx z) \leq (x \approx z)$ (transitivity), and $(x \approx y) = 1$ implies $x = y$. We say that a binary **L**-relation $R \in L^{X \times Y}$ is *compatible* with respect to $\approx_X$ and $\approx_Y$ if $R(x, y) \otimes (x \approx_X x') \otimes (y \approx_Y y') \leq R(x', y')$ for any $x, x' \in X$, $y, y' \in Y$. Analogously an **L**-set $A \in L^X$ is *compatible* with respect to $\approx_X$ if $A(x) \otimes (x \approx_X x') \leq A(x')$ for any $x, x' \in X$. Given $A, B \in L^X$, in agreement with [5] we define the subsethood degree $\mathrm{S}(A, B)$ of A in B by $\mathrm{S}(A, B) = \bigwedge_{x \in X} A(x) \rightarrow B(x)$. For $A \in L^X$ and $a \in L$, the set ${}^aA = \{x \in X;\ A(x) \geq a\}$ is called the a-cut of A. Analogously, for $R \in L^{X \times Y}$ and $a \in L$, we denote ${}^aR = \{(x, y) \in X \times Y;\ R(x, y) \geq a\}$. For $x \in X$ and $a \in L$, $\{{}^a/_x\}$ is the **L**-set defined by $\{{}^a/_x\}(x) = a$ and $\{{}^a/_x\}(y) = 0$ for $y \neq x$.

Definition 2. *An* **L**-*order on a set X with an* **L**-*equality $\approx$ is a binary* **L**-*relation $\preceq$ which is compatible with respect to $\approx$ and satisfies the following conditions for all $x, y, z \in X$:*

$$(x \preceq x) = 1 \qquad (\textit{reflexivity}),$$

$$(x \preceq y) \otimes (y \preceq x) \leq (x \approx y) \qquad (\textit{antisymmetry}),$$

$$(x \preceq y) \otimes (y \preceq z) \leq (x \preceq z) \qquad (\textit{transitivity}).$$

(Cf. T-E-ordering from [4].) Since in residuated lattices $x \otimes y \leq x \wedge y$ for every $x, y \in X$, our relation is more general than **L**-order of [3] or [2] where antisymmetry is expressed by condition $(x \preceq y) \wedge (y \preceq x) \leq (x \approx y)$.

Note that

$$(x \preceq y) \otimes (y \preceq x) \leq (x \approx y) \leq (x \preceq y) \wedge (y \preceq x). \tag{1}$$

Indeed, the first inequality represents antisymmetry and the second one follows from compatibility (see proof of Lemma 4 in [3]):
$(x \approx y) = (x \preceq x) \otimes (x \approx x) \otimes (x \approx y) \leq (x \preceq y)$ and analogously, $(x \approx y) \leq (y \preceq x)$.

The inequalities (1) represent the fact that **L**-equality must satisfy the interval confinement as follows:

$$(x \approx y) \in [(x \preceq y) \otimes (y \preceq x),\ (x \preceq y) \wedge (y \preceq x)]\,.$$

On the other hand, the definition of **L**-order by [3] (which must satisfy the condition $(x \preceq_{[3]} y) \wedge (y \preceq_{[3]} x) \leq (x \approx y)$) leads to firm binding of **L**-equality with the upper bound of previous interval (see Lemma 4 of [3]), i.e.

$$(x \approx y) = (x \preceq_{[3]} y) \wedge (y \preceq_{[3]} x).$$

Now, we can interpret the relationship between **L**-order and **L**-equality defined either in [3] and in this paper as follows. By the definition, **L**-order is dependent on a given **L**-equality. However, if we change point of view and have a look to the inverse "dependence", we can see that

- by [3], an **L**-equality is binded with any corresponding **L**-order firmly,
- in our paper, an **L**-equality has certain freedom (with respect to a corresponding **L**-order).

This will play an important role during generalizing results achieved in [3].

If $\preceq$ is an **L**-order on a set X with an **L**-equality $\approx$, we call the pair $\mathbf{X} = \langle\langle X, \approx\rangle, \preceq\rangle$ an **L**-*ordered set*. In agreement with [3], we say that **L**-ordered sets $\langle\langle X, \approx_X\rangle, \preceq_X\rangle$ and $\langle\langle Y, \approx_Y\rangle, \preceq_Y\rangle$ are *isomorphic* if there is a bijective mapping $h : X \to Y$ such that $(x \approx_X x') = (h(x) \approx_Y h(x'))$ and $(x \preceq_X x') = (h(x) \preceq_Y h(x'))$ hold for any $x, x' \in X$.

Note that in case of firm binding of **L**-equality and **L**-order by [3] (see the note above), preservation of the **L**-order by the bijection h implies also preservation of the **L**-equality. Clearly this is not true for **L**-order defined in this paper.

3 Some properties of fuzzy ordered sets

In this section, we describe some notions and properties related to fuzzy ordered sets which represent appropriate generalizations of notions and facts known from classical (partial) ordered sets. These generalizations were introduced mainly in [2] and [3] (the fact that originally they used less general definition of **L**-order is unimportant).

Definition 3. *For an* **L**-*ordered set* $\langle\langle X, \approx\rangle, \preceq\rangle$ *and* $A \in L^X$ *we define the* **L**-*sets* $\mathcal{L}(A)$ *and* $\mathcal{U}(A)$ *in* X *by*

$$\mathcal{L}(A)(x) = \bigwedge_{x' \in X} (A(x') \to (x \preceq x')) \text{ for all } x \in X,$$

$$\mathcal{U}(A)(x) = \bigwedge_{x'' \in X} (A(x'') \to (x'' \preceq x)) \text{ for all } x \in X.$$

$\mathcal{L}(A)$ *and* $\mathcal{U}(A)$ *are called the lower cone and upper cone of* A*, respectively.*

These **L**-sets can be described as the **L**-sets of elements which are smaller (greater) than all elements of A. We will abbreviate $\mathcal{U}(\mathcal{L}(A))$ by $\mathcal{UL}(A)$, $\mathcal{L}(\mathcal{U}(A))$ by $\mathcal{LU}(A)$ etc.

Definition 4. *For an* $\mathbf{L}$*-ordered set* $\langle\langle X, \approx\rangle, \preceq\rangle$ *and* $A \in L^X$ *we define the* $\mathbf{L}$*-sets* $\inf(A)$ *and* $\sup(A)$ *in* X *by*

$$(\inf(A))(x) = (\mathcal{L}(A))(x) \wedge (\mathcal{UL}(A))(x) \text{ for all } x \in X,$$

$$(\sup(A))(x) = (\mathcal{U}(A))(x) \wedge (\mathcal{LU}(A))(x) \text{ for all } x \in X.$$

$\inf(A)$ *and* $\sup(A)$ *are called the infimum and supremum of* A*, respectively.*

Lemma 1. *Let* $\langle\langle X, \approx\rangle, \preceq\rangle$ *be an* $\mathbf{L}$*-ordered set,* $A \in L^X$*. If* $(\inf(A))(x) = 1$ *and* $(\inf(A))(y) = 1$ *then* $x = y$ *(and similarly for* $\sup(A)$*).*

Proof. The proof is almost verbatim repetition of the proof of Lemma 9 in [3].□

Lemma 2. *For an* $\mathbf{L}$*-ordered set* $\langle\langle X, \approx\rangle, \preceq\rangle$ *and* $A \in L^X$*, the* $\mathbf{L}$*-sets* $\inf(A)$ *and* $\sup(A)$ *are compatible with respect to* $\approx$*.*

Proof. The proof can be found in [2], namely in more general proof of Lemma 5.39 with regard to Remark 5.40. □

Definition 5. *For a set* X *with an* $\mathbf{L}$*-equality* $\approx$*, an* $\mathbf{L}$*-set* $A \in L^X$ *is called an* S*-singleton if it is compatible with respect to* $\approx$ *and there is some* $x_0 \in X$ *such that* $A(x_0) = 1$ *and* $A(x) < 1$ *for* $x \neq x_0$*.*

Remark 1. There are various definitions of fuzzy singletons (see e.g. [8] or [2]). Our definition represents the simplest one, that is why we call it S-singleton. Demanding more conditions than stated would lead to serious troubles in proof of Theorem 2. Note that in case of $\mathbf{L}$ equal to the Boolean algebra $\mathbf{2}$ of classical logic with the support $\{0, 1\}$, S-singletons represent classical one-element sets.

Lemma 3. *For an* $\mathbf{L}$*-ordered set* $\langle\langle X, \approx\rangle, \preceq\rangle$ *and* $A \in L^X$*, if* $(\inf(A))(x_0) = 1$ *for some* $x_0 \in X$ *then* $\inf(A)$ *is an* S*-singleton. The same is true for suprema.*

Proof. The assertion immediately follows from Lemmata 1 and 2. □

Definition 6. *An* $\mathbf{L}$*-ordered set* $\langle\langle X, \approx\rangle, \preceq\rangle$ *is said to be completely lattice* $\mathbf{L}$*-ordered if for any* $A \in L^X$ *both* $\inf(A)$ *and* $\sup(A)$ *are* S*-singletons.*

Theorem 1. *For an* $\mathbf{L}$*-ordered set* $\mathbf{X} = \langle\langle X, \approx\rangle, \preceq\rangle$*, the relation* ${}^1\!\preceq$ *is an order on* X*. Moreover, if* $\mathbf{X}$ *is completely lattice* $\mathbf{L}$*-ordered then* ${}^1\!\preceq$ *is a lattice order on* X*.*

Proof. The proof is analogous to the proof of Theorem 13 in [3]. □

4 Fuzzy concept lattices

We remind some basic facts about concept lattices in fuzzy setting. A formal **L**-*context* is a tripple $\langle X, Y, I\rangle$ where I is an **L**-relation between the sets X and Y (with elements called objects and attributes, respectively). For any **L**-context we can generalize notions introduced in Definition 3 as follows. Let X, Y be sets with **L**-equalities $\approx_X, \approx_Y$, respectively; $I \in L^{X\times Y}$ be an **L**-relation compatible with respect to $\approx_X$ and $\approx_Y$. For any $A \in L^X$, $B \in L^Y$, we define $A^\uparrow \in L^Y$, $B^\downarrow \in L^X$ (see e.g. [1]) by

$$A^\uparrow(y) = \bigwedge_{x\in X} (A(x) \to I(x, y)) \text{ for all } y \in Y,$$

$$B^\downarrow(x) = \bigwedge_{y\in Y} (B(y) \to I(x, y)) \text{ for all } x \in X.$$

Clearly, $A^\uparrow(y)$ describes the truth degree, to which "for each x from A, x and y are in I", and similarly $B^\downarrow(x)$. We will abbreviate $(A^\uparrow)^\downarrow$ by $A^{\uparrow\downarrow}$, $(B^\downarrow)^\uparrow$ by $B^{\downarrow\uparrow}$ etc. The equation $A^\uparrow = A^{\uparrow\downarrow\uparrow}$ holds true for all $A \in L^X$ (see e.g. [1]). Note that if $X = Y$ and $I = \preceq$ is an **L**-order on X, then $A^\uparrow$ coincides with $\mathcal{U}(A)$ and $B^\downarrow$ coincides with $\mathcal{L}(B)$. An **L**-*concept* in a given **L**-context $\langle X, Y, I\rangle$ is any pair $\langle A, B\rangle$ of $A \in L^X$ and $B \in L^Y$ such that $A^\uparrow = B$ and $B^\downarrow = A$ (see [2]).

We denote by $\mathcal{B}(X, Y, I)$ the set of all **L**-concepts given by an **L**-context $\langle X, Y, I\rangle$, i.e.

$$\mathcal{B}(X, Y, I) = \{\langle A, B\rangle \in L^X \times L^Y;\ A^\uparrow = B,\ B^\downarrow = A\}.$$

For any $\mathcal{B}(X, Y, I)$, we put (see [3])

$$(\langle A_1, B_1\rangle \preceq_S \langle A_2, B_2\rangle) = \mathrm{S}(A_1, A_2) \text{ for all } \langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in \mathcal{B}(X, Y, I).$$

The **L**-relation $\preceq_S$ obviously satisfies the conditions of reflexivity and transitivity. As to the antisymmetry, we need an **L**-equality. Therefore, consider an arbitrary **L**-equality $\approx$ on the set $\mathcal{B}(X, Y, I)$ such that $\preceq_S$ is compatible with respect to $\approx$ and the inequality

$$(\langle A_1, B_1\rangle \preceq_S \langle A_2, B_2\rangle) \otimes (\langle A_2, B_2\rangle \preceq_S \langle A_1, B_1\rangle) \le (\langle A_1, B_1\rangle \approx \langle A_2, B_2\rangle)$$

holds true for every $\langle A_i, B_i\rangle \in \mathcal{B}(X, Y, I),\ i \in \{1, 2\}$. (Existence of such an **L**-equality is demonstrated e.g. by $(\langle A_1, B_1\rangle \approx \langle A_2, B_2\rangle) = \mathrm{S}(A_1, A_2) \wedge \mathrm{S}(A_2, A_1)$ in [3].) Consequently, $\preceq_S$ is an **L**-order on $\langle \mathcal{B}(X, Y, I), \approx\rangle$ and we get an **L**-ordered set $\langle\langle \mathcal{B}(X, Y, I), \approx\rangle, \preceq_S\rangle$ which will act in further two theorems. Note that the **L**-ordered set is more general than **L**-concept lattice $\langle\langle \mathcal{B}(X, Y, I), \approx\rangle, \preceq_S\rangle$ of [3] because of more general **L**-equality.

The next theorem characterizing **L**-concept lattices needs further denotation. As usual, for an **L**-set A in U and $a \in L$, we denote by $a \otimes A$ the **L**-set such that

$(a \otimes A)(u) = a \otimes A(u)$ for all $u \in U$. If $\mathcal{M}$ is an **L**-set in Y and each $y \in Y$ is an **L**-set in X, we define the **L**-set $\bigcup \mathcal{M}$ in X (see [3]) by

$$(\bigcup \mathcal{M})(x) = \bigvee_{A \in Y} \mathcal{M}(A) \otimes A(x) \quad \text{for all } x \in X.$$

Clearly, $\bigcup \mathcal{M}$ represents a generalization of a union of a system of sets. For an **L**-set $\mathcal{M}$ in $\mathcal{B}(X,Y,I)$, we put $\bigcup_X \mathcal{M} = \bigcup \mathrm{pr}_X(\mathcal{M})$, $\bigcup_Y \mathcal{M} = \bigcup \mathrm{pr}_Y(\mathcal{M})$ where $\mathrm{pr}_X(\mathcal{M})$ is an **L**-set in the set $\{A \in L^X;\ A = A^{\uparrow\downarrow}\}$ of all extents of $\mathcal{B}(X,Y,I)$ defined by $(\mathrm{pr}_X\mathcal{M})(A) = \mathcal{M}(A, A^{\uparrow})$ and, similarly, $\mathrm{pr}_Y(\mathcal{M})$ is an **L**-set in the set $\{B \in L^Y;\ B = B^{\downarrow\uparrow}\}$ of all intents of $\mathcal{B}(X,Y,I)$ defined by $(\mathrm{pr}_Y\mathcal{M})(B) = \mathcal{M}(B^{\downarrow}, B)$. Hence, $\bigcup_X \mathcal{M}$ is the "union of all extents from $\mathcal{M}$" and $\bigcup_Y \mathcal{M}$ is the "union of all intents from $\mathcal{M}$" (see [3]).

Theorem 2. *Let $\langle X,Y,I\rangle$ be an* **L***-context. An* **L***-ordered set $\langle\langle \mathcal{B}(X,Y,I), \approx\rangle, \preceq_S\rangle$ is completely lattice* **L***-ordered set in which infima and suprema can be described as follows: for an* **L***-set $\mathcal{M}$ in $\mathcal{B}(X,Y,I)$ we have*

$$^1\mathrm{inf}(\mathcal{M}) = \left\{ \left\langle (\bigcup_Y \mathcal{M})^{\downarrow}, (\bigcup_Y \mathcal{M})^{\downarrow\uparrow} \right\rangle \right\} \tag{2}$$

$$^1\mathrm{sup}(\mathcal{M}) = \left\{ \left\langle (\bigcup_X \mathcal{M})^{\uparrow\downarrow}, (\bigcup_X \mathcal{M})^{\uparrow} \right\rangle \right\} \tag{3}$$

Proof. The proof of (2) and (3) is analogous to the proof of part (i) of Theorem 14 in [3] where differently defined antisymmetry is not used anywhere. Furthermore by Lemma 3, each **L**-ordered set $\langle\langle \mathcal{B}(X,Y,I), \approx\rangle, \preceq_S\rangle$ is completely lattice **L**-ordered. □

For any completely lattice **L**-ordered set $\mathbf{X} = \langle\langle X, \approx\rangle, \preceq\rangle$, a subset $K \subseteq X$ is called $\{0,1\}$-*infimally dense* ($\{0,1\}$-*supremally dense*) *in* $\mathbf{X}$ (cf. [3]) if for each $x \in X$ there is some $K' \subseteq K$ such that $x = \bigwedge K'$ ($x = \bigvee K'$). Here $\bigwedge$ ($\bigvee$) means infimum (supremum) with respect to the 1-cut of $\preceq$.

Theorem 3. *Let $\langle X,Y,I\rangle$ be an* **L***-context. A completely lattice* **L***-ordered set $\mathbf{V} = \langle\langle V, \approx_V\rangle, \preceq\rangle$ is isomorphic to an* **L***-ordered set $\langle\langle \mathcal{B}(X,Y,I), \approx\rangle, \preceq_S\rangle$ iff there are mappings $\gamma : X \times L \to V,\ \mu : Y \times L \to V$, such that*

(i) $\gamma(X \times L)$ *is* $\{0,1\}$*-supremally dense in* $\mathbf{V}$*,*

(ii) $\mu(Y \times L)$ *is* $\{0,1\}$*-infimally dense in* $\mathbf{V}$*,*

(iii) $((a \otimes b) \to I(x,y)) = (\gamma(x,a) \preceq \mu(y,b))$ *for all* $x \in X,\ y \in Y,\ a,b \in L$*.*

(iv) $(\langle A_1,B_1\rangle \approx \langle A_2,B_2\rangle) = \left(\bigvee_{x\in X} \gamma(x, A_1(x)) \approx_V \bigvee_{x\in X} \gamma(x, A_2(x))\right)$ *for all* $\langle A_1,B_1\rangle, \langle A_2,B_2\rangle \in \mathcal{B}(X,Y,I)$*.*

Proof. Let γ and μ with the above properties exist. If we define the mapping $\varphi : \mathcal{B}(X,Y,I) \to V$ by $\varphi(A,B) = \bigvee_{x\in X} \gamma(x, A(x))$ for all $\langle A,B\rangle \in \mathcal{B}(X,Y,I)$, then by the proof of Part (ii) of Theorem 14 in [3] (where differently defined antisymmetry is not used anywhere) the mapping φ is bijective and preserves fuzzy order. Thus, we have to prove that it preserves also fuzzy equality. However this is immediate:

$$(\varphi(A_1,B_1) \approx_V \varphi(A_2,B_2)) = (\bigvee_{x\in X} \gamma(x, A_1(x)) \approx_V \bigvee_{x\in X} \gamma(x, A_2(x))) =$$
$$= (\langle A_1,B_1\rangle \approx \langle A_2,B_2\rangle).$$

Conversely, let $\mathbf{V}$ and $\langle\langle \mathcal{B}(X,Y,I),\approx\rangle,\preceq_S\rangle$ be isomorphic. Similarly to [3], it suffices to prove existence of mappings γ,μ with desired properties for $\mathbf{V} = \langle\langle \mathcal{B}(X,Y,I),\approx\rangle,\preceq_S\rangle$ and for identity on $\langle\langle \mathcal{B}(X,Y,I),\approx\rangle,\preceq_S\rangle$ which is obviously an isomorphism. The reason for this simplification lies in the fact that for the general case $\mathbf{V} \cong \langle\langle \mathcal{B}(X,Y,I),\approx\rangle,\preceq_S\rangle$ one can take $\gamma\circ\varphi : X \times L \to V,\ \mu\circ\varphi : Y \times L \to V$, where φ is the isomorphism of $\langle\langle \mathcal{B}(X,Y,I),\approx\rangle,\preceq_S\rangle$ onto $\mathbf{V}$. If we define $\gamma : X \times L \to \mathcal{B}(X,Y,I),\ \mu : Y \times L \to \mathcal{B}(X,Y,I)$ by

$$\gamma(x,a) = \left\langle \{{}^{a}/_{x}\}^{\uparrow\downarrow}, \{{}^{a}/_{x}\}^{\uparrow} \right\rangle,$$

$$\mu(y,b) = \left\langle \left\{{}^{b}/_{y}\right\}^{\downarrow}, \left\{{}^{b}/_{y}\right\}^{\downarrow\uparrow} \right\rangle$$

for all $x \in X,\ y \in Y,\ a,b \in L$, then by the proof of Part (ii) of Theorem 14 in [3] (where differently defined antisymmetry is not used anywhere) these mappings γ,μ satisfy conditions (i–iii) of our theorem. So, it remains to prove condition (iv), i.e. the equality

$$(\langle A_1,B_1\rangle \approx \langle A_2,B_2\rangle) = \left(\bigvee_{x\in X} \gamma(x, A_1(x)) \approx_V \bigvee_{x\in X} \gamma(x, A_2(x))\right).$$

Since we consider identity on $\langle\langle \mathcal{B}(X,Y,I),\approx\rangle,\preceq_S\rangle$, we have $\approx = \approx_V$ and it suffices to prove that $\bigvee_{x\in X} \gamma(x, A(x)) = \langle A,B\rangle$ for all $\langle A,B\rangle \in \mathcal{B}(X,Y,I)$.

We start with proof of the equation $A = \bigcup_{x\in X} \left\{{}^{A(x)}/_{x}\right\}^{\uparrow\downarrow}$ for any $A = A^{\uparrow\downarrow}$. On the one hand we get for each $x' \in X$:

$$\left(\bigcup_{x\in X} \left\{{}^{A(x)}/_{x}\right\}^{\uparrow\downarrow}\right)(x') = \bigvee_{x\in X} \left\{{}^{A(x)}/_{x}\right\}^{\uparrow\downarrow}(x') =$$

$$= \bigvee_{x\in X} \left[\bigwedge_{y\in Y} \left\{{}^{A(x)}/_{x}\right\}^{\uparrow}(y) \to I(x',y)\right] =$$

$$= \bigvee_{x\in X} \left[\bigwedge_{y\in Y} \left(\bigwedge_{\tilde{x}\in X} \left\{{}^{A(x)}/_{x}\right\}(\tilde{x}) \to I(\tilde{x},y)\right) \to I(x',y)\right] =$$

$$= \bigvee_{x \in X} \left[\bigwedge_{y \in Y} (A(x) \to I(x,y)) \to I(x',y) \right] \geq$$

$$\geq \bigwedge_{y \in Y} [(A(x') \to I(x',y)) \to I(x',y)] \geq$$

$$\geq \bigwedge_{y \in Y} A(x') = A(x').$$

On the other hand we have:

$$\left(\bigcup_{x \in X} \left\{ {}^{A(x)}/_{x} \right\}^{\uparrow\downarrow} \right)(x') = \bigvee_{x \in X} \left[\bigwedge_{y \in Y} (A(x) \to I(x,y)) \to I(x',y) \right] \leq$$

$$\leq \bigvee_{x \in X} \left[\bigwedge_{y \in Y} \left(\bigwedge_{\tilde{x} \in X} A(\tilde{x}) \to I(\tilde{x},y) \right) \to I(x',y) \right] =$$

$$= \bigvee_{x \in X} \left[\bigwedge_{y \in Y} A^{\uparrow}(y) \to I(x',y) \right] =$$

$$= \bigvee_{x \in X} A^{\uparrow\downarrow}(x') = A^{\uparrow\downarrow}(x') = A(x').$$

Using also the definition of γ and Theorem 2, we obtain

$$\bigvee_{x \in X} \gamma(x, A(x)) = \bigvee_{x \in X} \left\langle \left\{ {}^{A(x)}/_{x} \right\}^{\uparrow\downarrow}, \left\{ {}^{A(x)}/_{x} \right\}^{\uparrow} \right\rangle =$$

$$= \left\langle \left(\bigcup_{x \in X} \left\{ {}^{A(x)}/_{x} \right\}^{\uparrow\downarrow} \right)^{\uparrow\downarrow}, \left(\bigcup_{x \in X} \left\{ {}^{A(x)}/_{x} \right\}^{\uparrow\downarrow} \right)^{\uparrow} \right\rangle =$$

$$= \langle A^{\uparrow\downarrow}, A^{\uparrow} \rangle = \langle A, B \rangle. \qquad \square$$

Remark 2. Note that the essential difference between Theorem 3 in this paper and Theorem 14, part (ii) in [3] lies in differently defined **L**-ordered sets (see the notes at the end of Section 2). Therefore in comparison to Theorem 14 of [3], Theorem 3 must contain "additional" condition (iv) which is necessary for isomorphism between (more general) **V** and $\langle\langle \mathcal{B}(X,Y,I), \approx\rangle, \preceq_S\rangle$.

5 Work in progress

There is an interesting proposition which deals with a completely lattice **L**-ordered set $\langle\langle \mathcal{B}(V,V,\preceq), \approx_S\rangle, \preceq_S\rangle$ such that

- $\preceq$ is an **L**-order on V,

- $\approx_S$ denotes an **L**-equality defined by $(\langle A_1, B_1\rangle \approx_S \langle A_2, B_2\rangle) = (v_1 \approx v_2)$ where v_i $(i \in \{1,2\})$ is the (unique) element of V such that $(\sup(A_i))(v_i) = 1$. (Thus fuzzy equality between **L**-concepts $\langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in \mathcal{B}(V, V, \preceq)$ is expressed by fuzzy equality between suprema of their extents.)

Proposition 1. *A completely lattice* **L***-ordered set* $\mathbf{V} = \langle\langle V, \approx_V\rangle, \preceq\rangle$ *is isomorphic to* $\langle\langle\mathcal{B}(V, V, \preceq), \approx_S\rangle, \preceq_S\rangle$.

The proposition represents a corollary of Theorem 3, but an elegant proof of this fact is a matter of further investigations.

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