A Novel Fuzzy Implication Stemming from a Fuzzy Lattice Inclusion Measure

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Abstract. We introduce a fuzzy implication stemming from a fuzzy lattice inclusion measure. We study "reasonable axioms" and properties of the aforementioned fuzzy implication, which (properties) are typicaly required in the literature and could be important in certain applications.

Key words: Fuzzy lattice theory, Fuzzy implications, Inclusion measure

1 Introduction

A number of basic properties of the classical (logic) implication have been generalized in fuzzy implications; hence, a number of "reasonable axioms" have been proposed tentatively for fuzzy implications [10], [12].

Lately, an *inclusion measure* function was introduced in a mathematical lattice $(\mathsf{L}, \leq)$ for fuzzying the corresponding (crisp) partial order relation [6], [8]. In this paper we study a fuzzy implication stemming from the aforementioned inclusion measure. We show that the proposed fuzzy implication satisfies most of the "reasonable axioms" proposed in the literature. We study additional properties of our proposed fuzzy implication. The latter properties are typicaly required in the literature and could be important in certain applications.

The layout is as follows. Section 2 presents mathematical preliminaries. Section 3 introduces a novel fuzzy implication. Section 4 concludes by summarizing the contribution including also a description of potential future work.

2 Mathematical Preliminaries

This section presents basic notions of (Fuzzy) Lattice Theory and Fuzzy Sets Theory [2], [3], [6], [8], [10], [11].

2.1 Inclusion Measure

Consider the following definitions.

Definition 1. *Given a set P, a binary relation $(\leq)$ on P is called* **partial order** *if and only if it satisfies the following conditions for all x, y, $z \in P$:*

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PO1. Reflexivity: $x \leq x$.
PO2. Antisymmetry: $x \leq y$ *and* $y \leq x \Rightarrow x{=}y$.
PO3. Transitivity: $x \leq y$ *and* $y \leq z \Rightarrow x \leq z$.

Condition $PO2$ can be replaced by the following equivalent condition:

$PO2'$. *Antisymmetry:* $x \leq y$ and $x \neq y \Rightarrow y \nleq x$.

Definition 2. *A* **partially ordered set***, or* **poset** *for short, is a pair* $(P, \leq)$*, where* P *is a set and* $\leq$ *is a partial order relation on* P.

Definition 3. *A* **(crisp) lattice** *is a poset* $(\mathsf{L}, \leq)$ *any two of whose elements have both a greatest lower bound and a least upper bound. A lattice* $(\mathsf{L}, \leq)$ *is called* **complete** *when each of its subsets* X *has both a greatest lower bound and a least upper bound in* L.

Definition 4. *Let* $(\mathsf{L}, \leq)$ *be a complete lattice with least and greatest elements* O *and* I*, respectively. An* **inclusion measure** *in* $(\mathsf{L}, \leq)$ *is a map* $\sigma : \mathsf{L} \times \mathsf{L} \rightarrow [0,1]$*, which satisfies the following conditions for* u*,* w*,* $x \in \mathsf{L}$*:*

IM0. $\sigma(x, O) = 0, \forall x \neq O$
IM1. $\sigma(x, x) = 1, \forall x \in \mathsf{L}$
IM2. $u \leq w \Rightarrow \sigma(x, u) \leq \sigma(x, w)$ *(Consistency Property)*
IM3. $x \wedge y < x \Rightarrow \sigma(x, y) < 1$

For a non-complete lattice condition $IM0$ is dropped.
Based on equivalence relation $x \wedge y < x \iff y < x \vee y$ ([1]) it follows that condition $IM3$ can, equivalently, be replaced by

$IM3'$. $y < x \vee y \Rightarrow \sigma(x, y) < 1$.

Conditions $IM1$ and $IM2$ imply $u \leq w \Rightarrow \sigma(u, u) \leq \sigma(u, w) \Rightarrow \sigma(u, w) = 1$, $u, w \in \mathsf{L}$. Hence, $\sigma(x, I) = 1$, $\forall x$ in a complete lattice $(\mathsf{L}, \leq)$.

2.2 Fuzzy Implications

Let X be a universe of discourse. A fuzzy set A in X ([14]) is defined as a set of ordered pairs $A = \{(x, \mu_A(x)) : x \in X\}$, where function $\mu_A : X \rightarrow [0,1]$ indicates the degree of membership of an element $x \in X$ in fuzzy set A.

Definition 5. *A binary operation* $i : [0,1] \times [0,1] \rightarrow [0,1]$*, is called* **fuzzy intersection** *if it is an extension of the classical Boolean intersection as follows*

BI1. $i(a, b) \in [0,1]$, $\forall a, b \in [0,1]$*, and*
BI2. $i(0,0) = i(0,1) = i(1,0) = 0$*;* $i(1,1) = 1$

A canonical model of fuzzy intersections is the family of *triangular norms*, or *t-norms* for short, defined rigorously next.

Definition 6. *A* **t-norm** *is a function* $i : [0,1] \times [0,1] \to [0,1]$*, which is commutative, associative, non-decreasing, and* $i(\alpha, 1) = \alpha$*,* $\forall \alpha \in [0,1]$*.*

A t-norm i is called *Archimedean* if it is both continuous and $a \in (0,1)$ implies $i(a,a) < a$; furthermore, a t-norm i is called *nilpotent* if it is both continuous and $\forall \alpha \in (0,1)$ there is a $\nu \in N$ such that $i(a, \ldots, a) = 0$. Archimedean norms are either *nilpotent* or *non-nilpotent*. The latter (norms) are also called *strict*.

Definition 7. *A function* $n : [0,1] \to [0,1]$ *is called* **negation** *if it is both non-increasing, i.e.* $n(a) \leq n(b)$ *for* $a \geq b$*, and* $n(0) = 1$*,* $n(1) = 0$*.*

A negation n is called *strict* if and only if n is both continuous and strictly decreasing, i.e. $n(a) < n(b)$ for $a > b$, $\forall a, b \in [0,1]$. A strict negation n is called *strong* if and only if it is *self-inverse*, i.e. $n(n(a)) = a$, $\forall a \in [0,1]$. The most popular strong negation is the *standard negation*: $n_S = 1 - a$.

A *triangular conorm*, or *t-conorm*, is a function $u : [0,1] \times [0,1] \to [0,1]$, which satisfies the following properties:

i) $u(a, 0) = a$, $\forall a \in [0,1]$,
ii) $u(a,b) \leq u(c,d)$ if both $a \leq c$ and $b \leq d$,
iii) $u(a,b) = u(b,a)$, $\forall a, b \in [0,1]$, and
iv) $u(u(a,b),c) = u(a, u(b,c))$, $\forall a, b, c \in [0,1]$.

A *fuzzy implication* is a function $g : [0,1] \times [0,1] \to [0,1]$, which for any truth values $a, b \in [0,1]$ of (fuzzy) propositions p, q, respectively, gives the truth value $g(a,b)$ of conditional proposition "if p then q". Function $g(.,.)$ should be an extension of the *classical implication* from domain $\{0,1\}$ to domain $[0,1]$.

The *implication operator* of classical logic is a map $m : \{0,1\} \times \{0,1\} \to \{0,1\}$ which satisfies the following conditions: $m(0,0) = m(0,1) = m(1,1) = 1$ and $m(1,0) = 0$. The latter conditions are typically the minimum requirements for a fuzzy implication operator. In other words, fuzzy implications are required to reduce to the classical implication when truth-values are restricted to 0 and 1; i.e. $g(0,0) = g(0,1) = g(1,1) = 1$ and $g(1,0) = 0$.

2.2.1 Properties of fuzzy implications

One way of defining an implication operator m in classical logic is using formula $m(a,b) = \overline{a} \vee b$, $a, b \in \{0,1\}$, where $\overline{a}$ denotes the negation of a. Another (equivalent) way of defining implication operator m in classical logic is using formula $m(a,b) = \max\{x \in \{0,1\} : a \wedge x \leq b\}$, $a, b \in \{0,1\}$.

Fuzzy logic extensions of the previous formulas, respectively, are

$$g(a,b) = u(n(a), b) \tag{1}$$

and

$$g(a,b) = \sup\{x \in [0,1] : i(a,x) \leq b\}, \tag{2}$$

$\forall a, b \in [0,1]$, where u, i and n denote a fuzzy union, a (continuous) fuzzy intersection, and a fuzzy negation, respectively. Note that functions u and i are *dual* (with respect to n) — Recall that a t-norm i and a t-conorm u are called *dual* (with respect to a fuzzy negation n) if and only if both $n(i(a,b)) = u(n(a), n(b))$ and $n(u(a,b)) = i(n(a), n(b))$ hold $\forall a, b \in [0,1]$.

Fuzzy implications obtained from (1) are usually referred to as *S-implications*, whereas fuzzy implications obtained from (2) are called *R-implications*.

Formula $m(a,b) = \overline{a} \vee b$ can also be rewritten, based on the law of *absorption of negation* in classical logic, as either $m(a,b) = \overline{a} \vee (a \wedge b)$ or $m(a,b) = (\overline{a} \wedge \overline{b}) \vee b$. Extensions of the latter equations in fuzzy logic are given, respectively, by

$$g(a,b) = u(n(a), i(a,b)) \tag{3}$$

and

$$g(a,b) = u(i(n(a), n(b)), b), \tag{4}$$

where u, i and n are required to satisfy the De Morgan laws. The fuzzy implications obtained from (3) are called *QL-implications* because they were originally introduced in quantum logic.

A number of basic properties of the classical (logic) implication have been generalized by fuzzy implications. Hence, a number of "reasonable axioms" emerged tentatively for fuzzy implications. Some of the aforementioned axioms are displayed next [10], [12].

A1. $a \le b \Rightarrow g(a,x) \ge g(b,x)$ *Monotonicity in first argument*
A2. $a \le b \Rightarrow g(x,a) \le g(x,b)$ *Monotonicity in second argument*
A3. $g(a, g(b,x)) = g(b, g(a,x))$ *Exchange Property*
This is a generalization of the equivalence between $a \Rightarrow (b \Rightarrow x)$ and $b \Rightarrow (a \Rightarrow x)$ in classical implication.
A4. $g(a,b) = g(n(b), n(a))$ *Contraposition*
A5. $g(1,b) = b$ *Neutrality of truth*
A6. $g(0,a) = 1$ *Dominance of falsity*
A7. $g(a,a) = 1$ *Identity*
A8. $g(a,b) = 1 \iff a \le b$ *Boundary Condition*
A9. g is a continuous function *Continuity*

We remark that one can easily prove that a *S-implication* fulfills axioms A1, A2, A3, A5, A6 and, when negation n is strong, it also fulfills axiom A4. Furthermore, a *R-implication* fulfills axioms A1, A2, A5, A6, and A7.

3 A Novel Fuzzy Implication

An inclusion measure (σ) can be used to quantify partial (fuzzy) set inclusion. In this sense $\sigma(x,y)$ is similar to alternative definitions proposed in the literature for quantifying a degree of inclusion of a (fuzzy) set into another one [6]. However, the aforementioned "alternative" definitions typically involve only overlapping (fuzzy) sets; otherwise, the corresponding inclusion index equals zero. Whereas, Definition 4 is more general and may give meaningful results in a general lattice — not only in the lattice of (fuzzy) sets. Indeed, $\sigma(x, y)$ can be interpreted as the fuzzy degree to which x is less than y; therefore notation $\sigma(x \leq y)$ is sometimes used in place of $\sigma(x, y)$.

Consider the following function in a lattice.

Definition 8. *A* **valuation** *in a lattice* $(\mathsf{L}, \leq)$ *is a real function* $v : \mathsf{L} \to \mathsf{R}$ *which satisfies* $v(x)+v(y) = v(x\wedge y)+v(x\vee y)$, $x, y \in \mathsf{L}$. *A valuation is called* **positive** *if and only if* $x < y$ *implies* $v(x) < v(y)$.

The following theorem shows an inclusion measure in a lattice based on a positive valuation function [6], [7].

Theorem 1. *If* $v : \mathsf{L} \to \mathsf{R}$ *is a positive valuation in a lattice* $(\mathsf{L}, \leq)$ *then function* $\sigma_\vee(x, y) = \frac{v(y)}{v(x\vee y)}$ *is an inclusion measure.*

In particular, for positive valuation function $v(x) = x$ inclusion measure $\sigma_\vee(x, y) = \frac{v(y)}{v(x\vee y)}$ equals $\sigma_\vee(x, y) = \frac{y}{x\vee y}$. The latter is a fuzzy implication because it reduces to the classical implication for truth values $x, y \in \{0, 1\}$; i.e. $\sigma_\vee(0, 0) = \sigma_\vee(0, 1) = \sigma_\vee(1, 1) = 1$ and $\sigma_\vee(1, 0) = 0$. Fig. 1 shows the graphical representation of fuzzy implication $\sigma_\vee$.

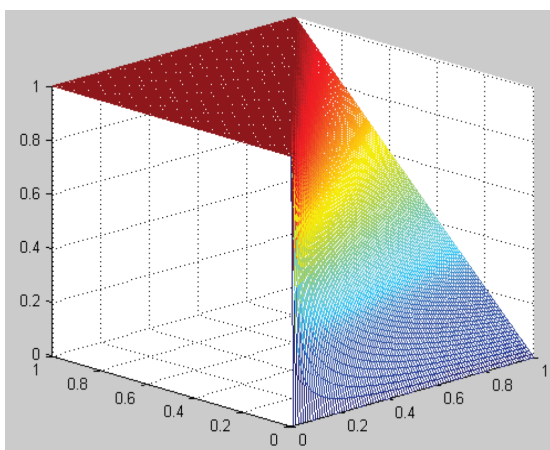

Fig. 1. Graphical representation of fuzzy implication $\sigma_\vee$

3.1 Properties of Fuzzy Implication $\sigma_\vee$

Fuzzy implication $\sigma_\vee$ satisfies the following "reasonable axioms" [10], [12] for fuzzy implications.

Proposition 1. *Consider fuzzy implication $\sigma_\vee$. Let $a, b, x \in [0,1]$. Then*

A1. $a \leq b \Rightarrow \sigma_\vee(a,x) \geq \sigma_\vee(b,x)$
A2. $a \leq b \Rightarrow \sigma_\vee(x,a) \leq \sigma_\vee(x,b)$
A3. $\sigma_\vee(a, \sigma_\vee(b,x)) = \sigma_\vee(b, \sigma_\vee(a,x))$
A4′. $\sigma_\vee(a,b) = \sigma_\vee(n(b), n(a))$ *— see in the following remark*
A5. $\sigma_\vee(1,b) = b$
A6. $\sigma_\vee(0,a) = 1$
A7. $\sigma_\vee(a,a) = 1$
A8. $\sigma_\vee(a,b) = 1 \iff a \leq b$
A9. $\sigma_\vee$ *is a continuous function*

We remark that axioms $(A1) - (A3)$ and $(A5) - (A9)$ in Proposition 1 can be proved immediately. For the standard fuzzy complement $n(a) = 1 - a$, axiom $(A4)$ holds only if $a \leq b$; whereas, for $a > b$, axiom $(A4)$ holds only if $a + b = 1$.

Additional "reasonable axioms" [13] include the following.

A10. $g(a, i(b,c)) = i(g(a,b), g(a,c))$
A11. $i(g(a,b), g(n(a),b)) = b$
A12. $i(g(0.5,b), g(0.5,b)) = b$
A13. $g(a, g(b,c)) = g(i(a,b),c)$

The next proposition shows how fuzzy implication $\sigma_\vee$ satisfies axioms *(A10)-(A13)* (with the "min" operator $(\wedge)$ employed as a fuzzy t-norm).

Proposition 2. *Consider fuzzy implication $\sigma_\vee$. Let $a, b, c \in [0,1]$. Then*

A10. $\sigma_\vee(a, (b \wedge c)) = \wedge(\sigma_\vee(a,b), \sigma_\vee(a,c))$
A11′. $\wedge(\sigma_\vee(a,b), \sigma_\vee(n(a),b)) > b$
A12′. $\wedge(\sigma_\vee(0.5,b), \sigma_\vee(0.5,b)) > b$
A13′. $\sigma_\vee(a, \sigma_\vee(b,c)) = \sigma_\vee(\wedge(a,b),c)$*, for* $c \geq a \wedge b$.

Proposition 2 can be proved immediately.

The following propositions describe some properties of fuzzy implication $\sigma_\vee$, which (properties) are often required in the literature because they could be important in certain applications [5], [10], [11].

Proposition 3. *Let us denote $\sigma_\vee(a,b)$ by $a \to b$. Then*

i) $(a \wedge b) \to c = (a \to c) \vee (b \to c)$
ii) $(a \vee b) \to c = (a \to c) \wedge (b \to c)$

Proposition 3 can be proved immediately.

Proposition 4. *Consider both fuzzy implication $\sigma_\vee$ and the standard negation $n_S = 1 - a$ (a' for short). Then*

i) $\sigma_\vee(a,1) = 1$, $\forall a \in [0,1]$
ii) $\sigma_\vee(\sigma_\vee(a,b),c) \leq \sigma_\vee(a, \sigma_\vee(b,c))$, $\forall a,b,c \in [0,1]$
iii) $\vee(a, \sigma_\vee(a,b)) = \sigma_\vee(a, \vee(b,c))$*, for* $a \leq b$

iv) $\wedge(a, \sigma_\vee(a,b)) \leq b$, *for* $a \leq b$
v) $\sigma_\vee(a,a') > a'$, $\forall a \in [0,1]$
vi) $\sigma_\vee(a',a) > a$, $\forall a \in [0,1]$
vii) $(\sigma_\vee(a,b))' \leq \sigma_\vee(a',b')$, $\forall a, b \in [0,1]$
viii) $\sigma_\vee(a,b) \geq b$, $\forall a, b \in [0,1]$
ix) $\sigma_\vee(a,b) \geq a'$, *for* $a \leq b$

Proposition 4 can be proved immediately.

Proposition 5. *Let us denote* $\sigma_\vee(a,b)$ *by* $a \to b$. *Then*

i) $(((a_1 \to a_2) \to a_3) \to \ldots) \to a_n = a_n$
when $a_1 \leq \ldots \leq a_n$ *and* n *is odd, and*
$(((a_1 \to a_2) \to a_3) \to \ldots) \to a_n = 1$
when $a_1 \leq \ldots \leq a_n$ *and* n *is even.*
ii) $(a_1 \to a_2) \to (a_2 \to a_3) \to \ldots \to (a_{n-1} \to a_n) = 1$
when $a_1 \leq \ldots \leq a_n$

Proposition 5 can be proved immediately.
Choosing the bounded sum, i.e. $a \oplus b = 1 \wedge (a+b)$, as a fuzzy union, it follows:

Proposition 6. *Let us denote* $\sigma_\vee(a,b)$ *by* $a \to b$. *Furthermore, let* $a, b \in [0,1]$. *Then*

i) $a \oplus (a \to b) = a \to (a \oplus b)$, *for* $a \leq b$
ii) $a \oplus (a \to b) \leq a \to (a \oplus b)$, *for* $a > b$

Proposition 6 can be proved immediately.

4 Discussion and Conclusion

This work has presented a novel fuzzy implication stemming from a fuzzy lattice inclusion measure function. It was shown that the proposed fuzzy implication satisfies a number of "reasonable axioms" and properties. Future work remains to study a number of interesting issues with a significant potential in practical applications as explained in the following.

A straightforward future extension includes consideration of intervals of truth-values in $[0,1]$ instead of the sole consideration of trivial intervals (single numbers) in $[0,1]$ — Note that the enabling technology for dealing with intervals was introduced recently [9]. An additional future extension regards consideration of L-fuzzy sets [4] towards a fuzzy implication involving granular (fuzzy) inputs [8].

Finally, note that using a different positive valuation function $v : \mathsf{L} \to \mathsf{R}$ than $v(x) = x$ is not expected to change any property because v is a strictly increasing function. We point out that a different inclusion measure function, namely $\sigma_\wedge(a,b) = \begin{cases} \frac{v(a \wedge b)}{v(a)}, & a \neq 0 \\ 1, & a = 0 \end{cases}$, [6], [7] is identical to inclusion measure $\sigma_\vee(a,b)$ under the assumptions of this work, i.e. $\sigma_\wedge(a,b) = \sigma_\vee(a,b)$.

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