What can lattices do for math. teaching & education?

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Abstract. A reasonable size example coming from cognitive psychology is re-analyzed with standard tools of FCA and Lattice Analysis. Developmental shifts / classifications are explored on a descriptive and graphical viewpoint, through attribute implications and unglued decomposition in regular intervals. This assesses child similarities in performance and behavior, while comparing intervals focuses on what children miss collectively to make further progress.

Keywords: (nested) line diagrams, (relative) canonical basis of implications, unglued decomposition, developmental shifts, construction of natural numbers.

Introduction

A first aim behind this paper is to address two topics of the conference *CLA'07* : "visualizing inherent properties in data sets" / "classifying systems based on relationships among objects and attributes through the concept of mathematical lattices" and to make this address concrete on a real example. In such a way the graphical outputs do not reduce to usual small models though staying still readable.

Secondly, to try convincing the researchers in educational / cognitive sciences that basic tools of Lattice Theory [Bi67] and Formal Concept Analysis (FCA, [GW99]) – which we usually mix together in French as *Analyse Latticielle* [D99]- can be most useful for describing their data and help in modeling underlying processes. All over the years, we have had many collaborations in applying Lattice Analysis to topics from the social and medical sciences (see [D99] and application papers quoted there) up to genetics (see [C&Al01], [Do&Al01], and [D&A01]), but not with that many cognitive scientists, although FCA has well spread in their close community of AI.

Third motivation, to try sharing our experience in visualizing lattice properties with our computer program (GLAD: General Lattice Analysis & Design, see [D83-96]). Instead of giving the program code which has been asked sometimes in conferences and surrounding communities -but would not make sense since it's now of an oldish system dependent conception- it seems to us far better to display convincing examples following carefully chosen features and to illustrate specifications through examples.

Last, as we did already twenty years ago with a paper attesting the usefulness of Lattice Theory and some of its standard tools to formalize and generate experimental designs (by characterizing them as partition sublattices see [D86]), it's a bow to the celebrated paper by G. Birkhoff: "What can lattices do for you?" ([Bi70]) which primed our own interest in applications of Lattice Theory to surrounding disciplines.

To follow these objectives, we will come back to a data set that Dr. Camilo Charron collected -and analyzed with statistical methods in his thesis (see [Cha98])-. This example can be classified between cognitive sciences / developmental & educational psychology. A small data subset has been extracted from a chapter devoted to “developmental shifts and knowledge transformations along the construction of natural and relative numbers” in children (4-14 years old).

Naturally, we will not enter deeply in the questions of semantics and interpretations, referring to the original work -or subsequent papers- to get more precise descriptions of the psychological and educational setting. We will follow a go-between attitude, linking separated topics and communities: to say enough about the content so that a user of FCA can understand the kind of specific questions encountered there, and reciprocally to illustrate and comment the tools and techniques with simple words so that psychologists can foresee the kind of drawbacks they could get after a small investment in papers on FCA giving them more abilities and insights.

Basic data and original questions

Out of the eight groups of children under study, only three are kept here (age 4.5 / 5.5 / 6.5 years old). They are -in French- denoted MS:”moyenne section”, GS:”grande section” and CP:”cours préparatoire”, the latter being the first year of elementary school, the others being the last two years of nursery school. Each group comprises 31 children, which passed a series of ten –“*à la Piaget*”- experiments to evaluate their mastering of operations and relations on natural numbers. Hence for each group of children the basic data consists in a binary table $C_{31}xA_{10}\rightarrow\{0,1\}$, for which (c,a)=1 whenever the child “c” masters attribute “a”, and (c,a)=0 otherwise. The ten attributes kept here concern only properties of *natural numbers*, operations and relations: for instance *order-ct* means mastering: [a>b & c=d implies a+c > b+d].

A:order	**F**:difference
B:equality	**G**:class-equiv.
C:order-ct	**H**:counting
D:equality-ct	**I**:identity-c
E:addition	**J**:commutativ.

(ct:conservation by translation, c:conservation)

Table 1. The attributes describing properties and operations on *natural numbers*.

As claimed in the thesis “The aim is to detect shifts [*ruptures*] of development along the construction of natural and relatives numbers…”, which is made precise later with additional hypothesis: “Child developments will be partially ordered, which will be assessed by child profiles [*patrons de réponses*] that will be genetically ordered and structured by exact [*as opposed to association rules*] implications. Implications will point out shift and / or knowledge transformations”. The main questions will be taken in charge naturally by *implications* which are one of the basic tools of FCA ([GW99 §2.3]), and these cognitive questions have an *intensional* nature. We will try to show that other natural questions relative to child classifications can also be raised, addressing more *extensional* –and educational- questions.

Lattice and analysis toolkit

The prerequisites for mastering the analysis and understanding the graphics comprise a few standard theorems and properties of FCA (see [GW99]) and applications of lattice analysis ([D99]). They will not be repeated here in mathematical terms, that would be useless and redundant. However they will be commented in everyday words, as a reminder and so that not-lattice-minded-users can foresee the content.

The basic data -for each group- can be seen as a binary relation $I \subseteq C_{31} x A_{10}$ which is represented by a so-called "*context*" ($C_{31}, A_{10}, I \subseteq C_{31} x A_{10}$) keeping tracks of the children and attribute labels, where $(c,a) \in I$ means children "c" masters attribute "a". For a subset $B \subseteq A_{10}$ of attributes let $B^{\downarrow} := \{c \in C_{31} / cIa \text{ all } a \in B\}$ be its *extension*. For a subset $D \subseteq C_{31}$ of children let $D^{\uparrow} := \{a \in A_{10} / dIa \text{ all } d \in D\}$ be its *intension*. As $B^{\downarrow\uparrow\downarrow} = B^{\downarrow}$ (all $B \subseteq A_{10}$) and $D^{\uparrow\downarrow\uparrow} = D^{\uparrow}$ (all $D \subseteq C_{31}$), the pair of maps $B \rightarrow B^{\downarrow}$ (all $B \subseteq A_{10}$) and $D \rightarrow D^{\uparrow}$ (all $D \subseteq C_{31}$) defines a *Galois connection* between the power sets of A_{10} and C_{31}, and two *closure operators* in them, that gives the matching extensions / intensions. In words, by the *Galois* / *concept lattice* construction the context is unfolded in a *concept system* which can be drawn, each *concept* being defined both in extension / intension, and the organization of concepts being driven upwards by the *join operation* (defined by intersection of intensions), and dually downwards by the *meet operation* (defined by intersection of child groups). These underlying mathematical structures are known since the first 1940 edition of [Bi67], and papers by O. Ore (see [O44]) and others.

Now a lattice can become really cumbersome and complex to draw, so that it is most often useful to *label* it *minimally* by locating each attribute $a \in A$ at $(a^{\downarrow}, a^{\downarrow\uparrow})$ i.e. the higher concept to which it belongs, and dually to locate each child $c \in C$ at $(c^{\uparrow\downarrow}, c^{\uparrow})$ that is the lowest concept having c in its extension. Pointing to any lattice element, its intension can be reconstructed by taking all attributes above it along the ordered lattice, and its extension comprises all children below it. A second drawing simplification introduced since the beginning of FCA [Wi82] is to start a lattice drawing for only a subset of attributes, and to introduce the remaining in a *nested line diagram* erasing lines parallel to those that are grafting these remaining attributes.

The duality between extensions / intensions is also carrying *implications* between attributes (symmetrically between children, that are meaningful in a *social network* perspective: who is together with whom?). When two attributes $a,b \in A$ are such that $a^{\downarrow} \subseteq b^{\downarrow}$ -so that a<b in the lattice- this can be read as a *simple* (premise) implication $a \rightarrow b$. When $(ab)^{\downarrow\uparrow} = abcd$, this indicates that $ab \rightarrow cd$ holds in the data. Both kind of implications can be deciphered graphically in the lattice. The simple ones will define the (*pre*)*order of attributes*, while the latter will be recognized by: the meet of a and b will "capture" c,d upwards. Fortunately, there is a *canonical basis of implications* summarizing all those holding in the data, which was the main result of [GD84-86] (see also [D84-87] for a more latticial version and [G84-87] for a nice algorithm).

Another procedure that will be used is the lattice *unglued decomposition* [GW99 §4.2]. The original idea came from classes of lattices encountered in Mathematics -*distributive*, *modular*- since they are decomposable in maximal *atomistic* intervals, a construction that have been generalized to arbitrary lattices using *tolerances* (i.e. *similarities* respecting lattice operations as lattice *congruences* do) and the lattice *cover* relation. In words, it is a way to look at the lattice "from further" by considering faithful similarities between attributes, and symmetrically upwards between children.

Intra-group results and analysis

For the first group ("MS", average age 4.5 years) the order of attributes –weighed with the % of the attribute's extension / the 31 children- is displayed in Fig. 1 (top left). It should be noted that three attributes are somehow "easy" for that group since they are already mastered by almost all children (*counting* 94%, *equality* 90%, *order* 84%). Many other attributes imply some of the latter, creating an attribute order of *length* one. In particular, the "most difficult" attributes for this group of children (*identity-c*, *commutativ.* 26%, and *difference* 6%) imply all three easy attributes.

A nested line diagram of the lattice is unfold (top right), along *commutativ.* and *difference* -meaning that the lines parallel to the one joining *commutativ.* to its upper cover have not been drawn for simplifying the drawing. First remark completing the fact that there are many order relationships / dependence between pair of attributes, the lattice has 53 elements, which is small as compared to the 2^{10}=1024 subsets of A_{10}, that would be observed if there were a complete independence of the 10 attributes.

In the powerset of attributes this means that the closure operator $B \mapsto B^{\downarrow\uparrow}$ (all $B \subseteq A$) that generates the intensions, at the same time generates a lot of equivalence and implications between conjunctions of attributes. This can be summarized by the canonical basis of implications, which is listed at left hand side below the order of attributes in Fig. 1, together with the identification number 1-21 and the extensions / supports. The implications are also grafted into the nested line diagram of the lattice, where they can be located by their identification number and intension, which is obtained from the list by taking the union of their premise and conclusion.

Hence for instance, the first simple implication (n°-6) at the list top, is $G \rightarrow AH$, and means that the 19 children mastering G=*class-equiv.* master A=*order* & H=*counting* as well. It can be located at the top right of the lattice and also deciphered in the lattice structure by the fact that the intension of the element labelled G is GAH, since collecting G and H along the lattice lines above G. As for non-simple implication with complex premises, for instance below implication n°-6 is located implication n°-7 -which reads $ACEH \rightarrow G$- indicates that the eight children that master A=*order* & C=*order-ct* & E=*addition* & H=*counting* master *class-equiv*.
In the basis list, the implications have been ranked by extension's decreasing order. Within the lattice, closer their intension is to the lattice top, bigger their intension is obviously. Dually, closer to the bottom they have smaller extensions. Now some implications could be understood as natural / obvious / "by construction". Thus, that D=*eqality-ct* $\rightarrow$ *B=equality* will not surprise anybody, even if their percentage extensions 58% / 90% may require some comments. Other implications may come from sampling questions: after all 31 children for exploring and assessing something definite on a universe with 2^{10} elements is not a lot… Hence, the basis of implications should be scanned thoroughly by the researcher with all these considerations in mind.

On a "macro level", the lattice is decomposable in six intervals (see Fig. 2) by unglued decomposition, which is quite a strong since rare property for arbitrary lattices. These intervals are ordered along a 2x3 product of chains. This gives a macro scaling of attributes of "similar difficulty level" –regarding that group of children-, with the easy attributes *counting-equality-order* to which is now added *class-equiv.* at the top, which are followed downwards by more difficult attributes scaled in two

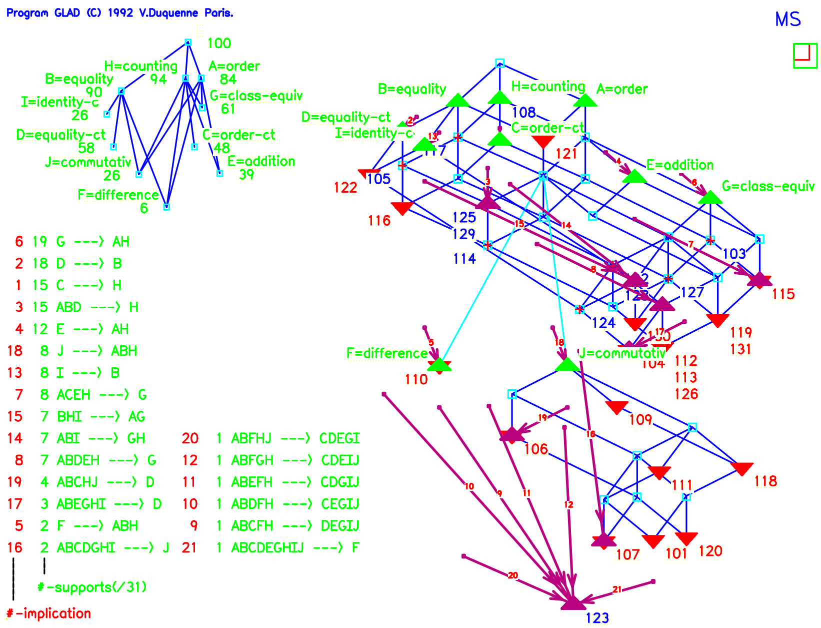

Fig. 1. Attribute order (top left) and nested line diagram of the lattice (right) along *difference* & *commutativ.*, with canonical basis of 21 implications and their supports.

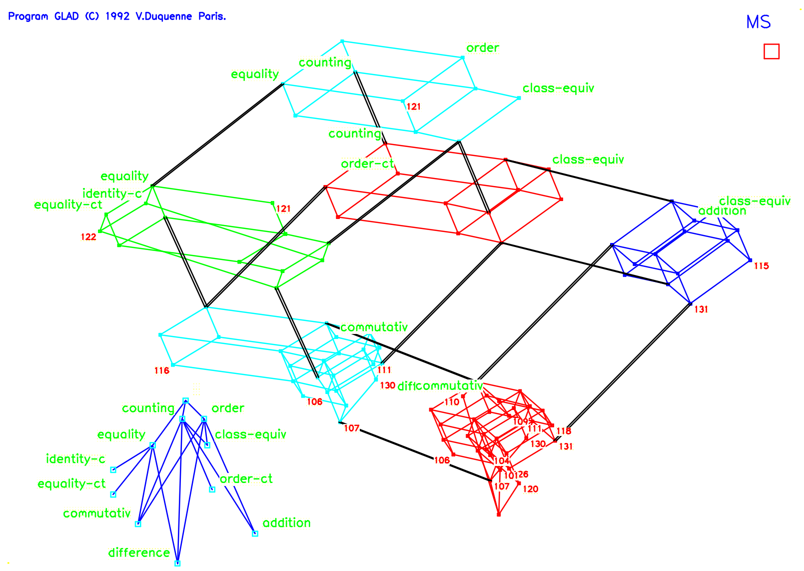

Fig. 2. Unglued decomposition in the direct product of 2x3 intervals. Each interval groups together the attributes of similar difficulties. Dually the children are grouped and scaled in homogeneous groups regarding their level of performance.

distinct directions, *order-ct* above *addition* on the one hand, and (*idendity-c*, *equality-ct*) on the other, which are themselves refined by *commutativ.* then *difference*. Notice that in that 2x3 direct product of intervals, *commutativ.* is below *order-ct*, while *difference* is below all intervals, hence with maximum difficulty level for that group. However this doesn't mean that *difference* is made comparable with *addition* in the lattice order relation: it is below in the 2x3 lattice of intervals i.e. for the "macro" difficulty levels that are made similar & scaled by the similarity (tolerance) relation.

The 2x3 interval product structures the (pre)order of children in terms of similar level of performance. The maximum element of the bottom interval is the lower cover of the element labeled 121 (close to the top) of which the intension is *ABH*, and extension comprises 23 out of the 31 children. This means that the five other intervals structure the nine remaining children, some of which being in difficulties: for instance child 121 (at the top) may require special attention and support, and the same is true for kids 115 / 103 (top right), or 122 / 105 / 116 / 177 (top left), but with different programs. An interesting outcome of this analysis could be to make proposals to constitute homogeneous groups of children of "similar level of performance".

A teacher aware of this information in real time could consequently define strategies for planning games and exercises for training the kids regarding their specific needs. Besides personalized cares as before, the teacher may want to train bigger groups of children, if possible made homogeneous. For example, the fifteen children that belong to the interval containing child 116 and *commutativ.* (bottom left, between child 107 and *equality* & *counting*) share the property of "not mastering *class-equiv.* and *addition*": all of them should benefit of special games / training specifically oriented to master these two attributes. Hence, looking within intervals reveals the specificity of performances for children of similar level, while comparing intervals can focus on what they miss or require to collectively make further progress.

For the second group ("GS", age 5.5 years) the global structure is quite different. *Counting* and *order* are now mastered by all the children, hence become superfluous regarding the lattice structure, and create eight obvious implications (expelled from the basis, see Fig 3, top left). There is only one simple implication left *identity-c* → *equality* & *class-equiv.*, a rather poor order structure between attributes, although the lattice is quite small (61 out of the potential 2^8=256 elements). Five attributes are mastered by more than 87% of the children, *commutativ.* and *difference* are no more under-represented, with now 48% and 58%, respectively. This contributes to the fact that the lattice has many elements below them: they associate in similar ways with other attributes (as opposed to what is occurring for "MS"-group), along nearly isomorphic intervals (ideals). Out of the twelve implications whose extensions are at least ten, after having removed superfluous A, H from their premises, nine have two attributes (ex: FG → B). As compared with the younger "MS"-group, it could be said that the general structure for the group of "GS"-children explodes in the direction of pairwise independence between attributes, almost without order relationships between them, but global independence and combinatorial explosion are tamed and reduced by a series of rather simple premise implications going in all directions. This explosion will perhaps not surprise teachers and parents who know that *grande-section*, which is the last year of *école maternelle* before the more academic universe of *école primaire*, is a special turn in the curriculum of children, a year of all discoveries…

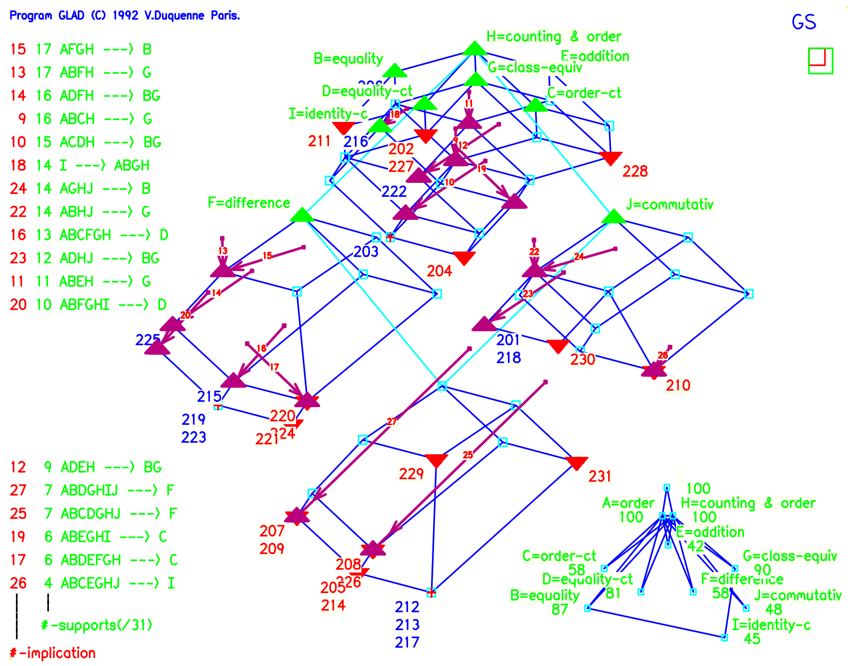

Fig. 3. Nested line diagram of the lattice (group 2: "GS"), with canonical basis of implications (18 out of 26, *counting* & *order* being confused with the top generate 8 obvious implications) and supports. Here, only $(I \rightarrow ABGH)$ has a simple premise.

As for the third group ("CP", age 6.5 years), 12 kids (out of 31) master all the attributes (to be compared with only one and three for the younger "MS" and "GS" groups, respectively), and most of the others are close -even very close- to the bottom. This is reassuring: a majority of these children now master almost all the 10 attributes. Moreover, five attributes receive more than 80%, while the others get between 58%-74%. After removing the three superfluous attributes (confused to the top with 100%), it remains only nine implications in the canonical basis, with only *difference* $\rightarrow$ *class-equiv.* $\rightarrow$ *order* and *commutativ.* $\rightarrow$ *order* which scale and order the attributes (see Fig. 4). These three simple implications control the lattice unglued decomposition downwards (see Fig. 5), in a chain of three totally ordered intervals.

At the top, there are only two children (321,326) who do not master *order* –and a fortiori the attributes implying *order*- and are somehow in difficulty: they should be helped specifically. The middle interval (between children 320,328 up to *order*) contains 10 children. They should be trained about *difference* to try collectively to master it and move down to the bottom interval. Downwards there are 18 children who all master *difference*, which is still the more difficult attribute for this older group. In this bottom interval, a small group of five kids (306,317,318,319,327) may have an interest in working first *commutativ*. As compared with "GS", this group is far from displaying independence, and the analysis shows that three subgroups could be profitably distinguished to be first trained on *order*, or *commutativ.* and *difference*.

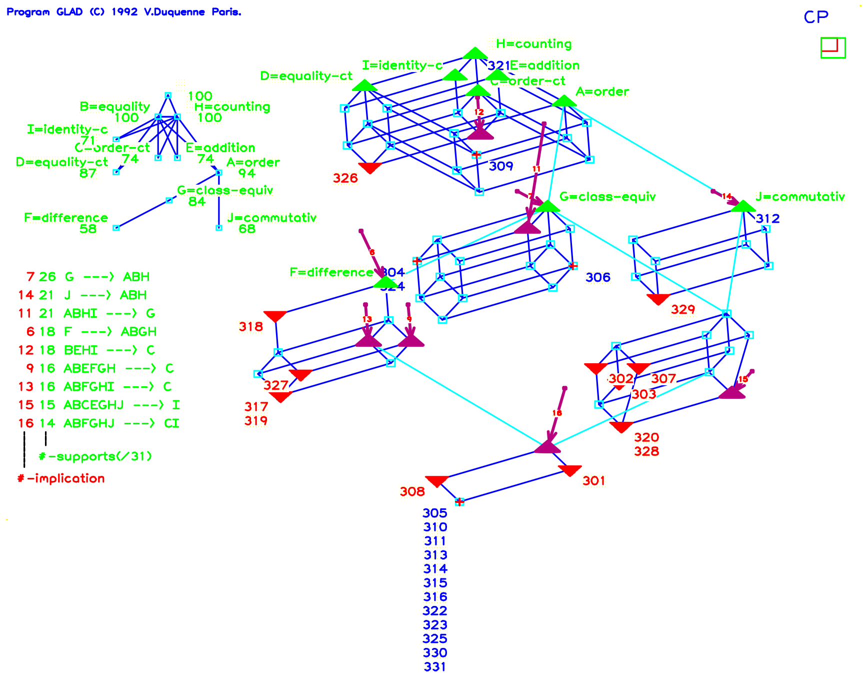

Fig. 4. Attribute preorder and the canonical basis of implications (9/16, *equality* & *counting* being confounded). There are only three simple implications.

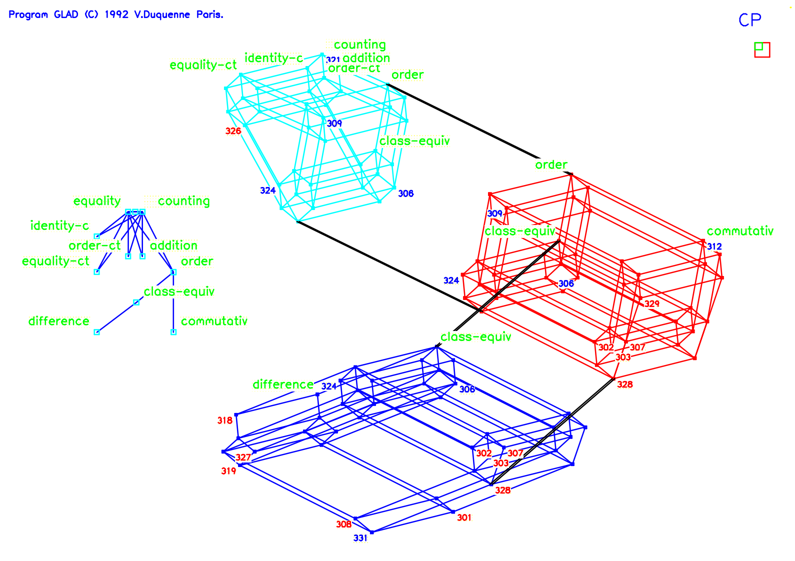

Fig. 5. Unglued decomposition in three intervals down-generated by the implications: *class-equiv*, *commutativ.* → *order* and *difference* → *class-equiv*, respectively.

Inter-group analysis and conclusion

The three groups of children have been put together in a $C_{93}xA_{10}$ table, for which the lattice L_{123} is represented by a nested line diagram in Fig. 6. As for the previous "CP" group, the lattice appears to be linearly unglued in three intervals, where the middle one –between child 204 to *counting* see Fig 7- comprises 35 children, and can be seen as down generated by *order-ct* (60%) and *addition* (52%). The top one is up generated by children 121,122 who do not master *counting*, while the bottom interval can be seen as down generated by the two challenging attributes *commutativ.* (47%) and *difference* (41%). This structure scales the children in three levels of performance: to master *counting* or not, and the same question with *commutativ.* and *difference*.

Now, it is interesting to scan through all children' labels to notice first that older children have a tendency to be located down in the lattice: they know more, fortunately... But to study this distribution of ages spread along the lattice may also give teachers the idea to organize small groups gathering together children of different forms, because they would have the same or homogeneous performances. Hence, the interval between child 204-*class-equiv.* comprises children of the three forms who all master *class-equiv.* but not *commutativ.* nor *difference*, so that they could benefit from activities to get them. This may help in organizing some "groupes de niveaux" (level groups) in a same school, in the direction of more open education.

Coming back to intensional questions, one may raise the following one: because of the institutional gap between *école maternelle* and *école primaire*, but also due to the kind of explosion that was stated for the "GS" group, could it be possible to summarize in a compact way the shift between groups 1+2 ("MS"+"GS") and group 3 ("CP")? A direct answer is available with the notion of *relative canonical basis of implications* (see [Do&Al01] and GLAD [83-96]) expressing how the lattice L_{123} –mixing together the three groups- is projected onto the lattice L_3 (which is a sub-semi-lattice of L_{123}), and that will be denoted by $B_{3/123}$. We have used this notion for years, specially in genetics. The relative basis $B_{3/123}$ is listed in Fig. 8 where each implication is weighed by its "123"-premise's extension and "3"-intension's extension, so that their difference shows how many "12"-children will be pushed down onto the 3-intension. The first two implications come partly from the fact, that B and H are superfluous with 100% among "3"-children, and assess (see Fig. 8) that respectively 55 and 60 "12"-children are "pushed" to BH from B and H. Similarly, 57 and 43 "12"-children are "pushed" to ABH and DBH. Many of the remaining implications contain ABH in their premise, so that BH can be erased to get simple premise, since A $\rightarrow$ BH. Hence the fifth implication may be simplified to A:*order* & F:*difference* $\rightarrow$ G:*class-equiv.* which concerns 19 children. Interestingly the five remaining implications in the basis have only C:*order-ct* or I:*indentity-c* –or their conjunction- as a conclusion. Thus, the gap between "MS"+"GS" and "CP" can be characterized with a few implications expressing either that *equality* and *counting* are fully mastered, or how *class-equiv.*, *order-ct* and *identity-c* are better mastered in "CP".

Before concluding, it should be added that the lattice L_{123} has a structural property which is interesting, even if it could not be seen easily but was discovered through a program: out of its 124 intensions, 114 have a unique *minimal generator*. In words, for any such intension $B^{\downarrow\uparrow} \subseteq A$ there is exactly one subset $Bo \subseteq B^{\downarrow\uparrow}$ such that $Bo^{\downarrow\uparrow} = B^{\downarrow\uparrow}$

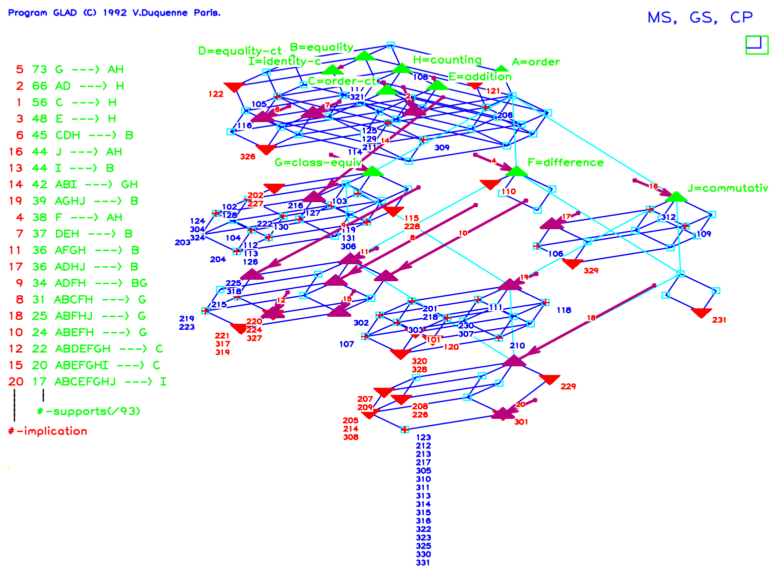

Fig. 6. Nested line diagram of the lattice (groups 1,2,3: 93 kids) with canonical basis of implications and their supports. A child's first digit indicates his / her group (1-3).

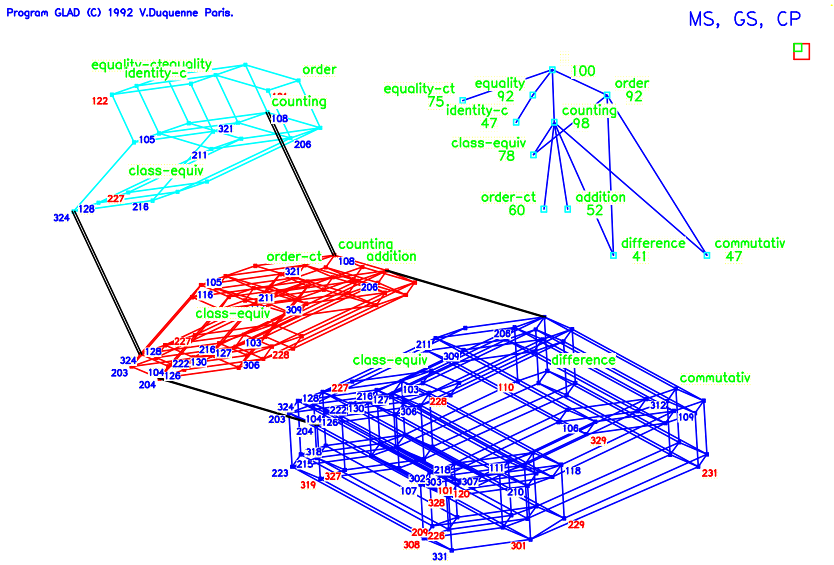

Fig. 7. Attribute order and lattice decomposition into three intervals, generated by *order-ct*, *addition* → *counting* and *difference*, *commutativ.* → *counting* & *order*.

and which is minimal for this property. Consequently, the lattice is “nearly” *join-distributive* (see GW99 p. 228), which means that collectively, the children’ performances behave “nearly” like if it was structured on an abstract *convexity space*. A lot of literature on convexity lattices has been written. They have many different characterizations (see for instance [M85]). In particular in such a convexity lattice, for any $Bo^{\downarrow\uparrow}$ and its unique minimal generator Bo, $(Bo^{\downarrow\uparrow}\backslash\{b\})$ is an intension for all “*extreme points*” b of Bo, so that the interval up-generated by the upper covers of $Bo^{\downarrow\uparrow}$ is Boolean. This expresses a property of *local independence* of extreme points. In other words the implication $Bo \rightarrow Bo^{\downarrow\uparrow}$ indicates that the set Bo of “extreme points of $Bo^{\downarrow\uparrow}$” implies what is collected “inside” their *hull* for the convexity structure.

The convexity structure provides a model for these three group behavior, but that can be also formulated as a model for each individual: a child $c \in C$ belongs to such or such a group’s extension -say $B^{\downarrow}$ -if and only if the children “c” masters a specific and unique set of keys (“extreme” attributes Bo). This provides a fine description of the minimal prerequisites which are sufficient although necessary for mastering these mathematical concepts and finding one’s path in the natural number construction.

Relative basis $B_{3/123}$

```
86 31 55 B ---> H
91 31 60 H ---> B
86 29 57 A ---> BH
70 27 43 D ---> BH
37 18 19 ABFH ---> G
27 18  9 BEHI ---> C
24 16  8 ABEFGH ---> C
27 16 11 ABFGHI ---> C
21 15  6 ABCEGHJ ---> I
25 14 11 ABFGHJ ---> CI
 |  #3 |
 |    (#123-#3)
 #123
```

Fig. 8. The relative canonical basis of implication $B_{3/123}$ summarizes how the elements of L_{123} are "pushed down" onto its (sub-semi)lattice L_3 in a minimal way. The difference of extents (#123-#3) indicates the number of such children (groups 1,2).

For concluding, this note re-analyzes data on the construction of natural numbers in children of 4.5-6.5 years old, with a descriptive method based on orders and lattices. The three groups behave quite differently in terms of attribute orders. While the “GS” order is almost an antichain of pairwise incomparable attributes, the two others -“MS” and “CP”- have a rich structures which are emphasized as they generate unglued decompositions of the lattices. Surprisingly the same properties apply to the lattice L_{123} that mixes together the three groups. This provides a scaling of the attribute orders in subsets of similar difficulties. Dually this structures the children in terms of homogeneous sub-groups, which would be most interesting for training the kids on specific target attributes. In the same manner a close examination of L_{123} can give rise to proposals for defining small groups mixing together children of different forms in the direction of a more open school. The shift between “MS+GS” to “CP” has been characterized through a relative basis of implications, and L_{123} behaves like convexity lattices that provides minimal keys as a model for describing child performance.

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References

[Bi67] Birkhoff G., *Lattice Theory*, Amer. Math. Soc. Colloq. Publ. Vol.25, third edition, Amer. Math. Soc., New York, N. Y. 1967 (first ed. 1940).

[Bi70] G. Birkhoff, What can lattices do for you?, in *Trends in Lattice Theory* (J. C. Abbott ed.), Van Nostrand, New York, 1970.

[C&Al01] Chabert C., Cherfouh A., Delabar J.-M. and Duquenne V., Assessing implications between genotypic / phenotypic variables through lattice analysis, *Behavior Genetics 31* (2001) 125-139.

[Cha98] Charron C., Ruptures et continuités dans la construction des nombres, Thesis (1998), Université Paris V-René Descartes, Paris.

[Do&Al01] Doyen A.-L., Duquenne V., Nuques S. and Carlier M., What can be learned from a lattice analysis of a laterality questionnaire?, *Behavior Genetics 31* (2001) 193-207.

[D84-87] Duquenne V., Contextual implications between attributes and some representation properties for finite lattices. in *Beitrage zur Begriffsanalyse*, (B. Ganter, R. Wille and K.E. Wolf eds), 1987, Mannheim: 213-239.

[D83-96] Duquenne V., GLAD (General Lattice Analysis & Design): a FORTRAN program for a glad user, *ORDAL 96*: *Order and decision-making* (I. Rival ed.), Ottawa, www.csi.uottawa.ca.

[D86] Duquenne V., What can lattices do for experimental designs?, *Mathematical Social Sciences 11* (1986) 243-281.

[D99] Duquenne V., Latticial structures in Data Analysis, *ORDAL 96*: *Order and decision-making* (I. Rival ed.), Ottawa, www.csi.uottawa.ca, published in *Theoretical Computer Science 217* (1999) 407-436.

[D&Al01] Duquenne V., Chabert C., Cherfouh A, Delabar J.-M., Doyen A.-L and Pickering D., Structuration of phenotypes / genotypes through Galois lattices and implications, in *Proc. of ICCS2001-CLKDD'01* (Stanford 07/2001, E. Mephu Nguifo et *al*. eds) 21-32, and *Applied Artificial Intelligence 17* (2003) 243-256.

[G84-87] Ganter B., Algorithmen zur Formalen Begriffsanalyse in *Beitrage zur Begriffsanalyse*, (B. Ganter ed.), 1987, Mannheim: 213-239 (preprint TH Darmstadt, 1984).

[GW99] Ganter B. and Wille R.. *Formal Concept Analysis, Mathematical Foundations.* Springer Verlag, Berlin, 1999.

[GD84-86] Guigues J.L. and Duquenne.V., Familles minimales d'implications informatives résultant d'un tableau de données binaires. *Mathematiques & Sciences Humaines 95* (1986) 5-18 (preprint Groupe Mathématiques et Psychologie, Université Paris V-René Descartes, 1984).

[M85] Monjardet B., A use of frequently rediscovering a concept, *Order 1* (1985) 415-417.

[O44] Ore O., Galois connexions, Trans. Amer. Math. Soc. *55* (1944) 493-513.]

[Wi82] Wille R., Restructuring lattice theory: an approach based on hierarchies of concepts. In *Ordered sets* (I. Rival ed), Reidel, Dordrecht-Boston (1982) 445-470.