A Categorial View at Generalized Concept Lattices: Generalized Chu Spaces*

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Abstract. We continue in the direction of the ideas from Zhang's paper [12] about a relationship between Chu spaces and Formal Concept Analysis. We modify this categorial point of view at a classical concept lattice to a generalized concept lattice (in the sense of [7]): We define generalized Chu spaces and show that together with (a special type of) their morphisms form a category. Moreover we define some interesting mappings and point on the commutativity of some diagrams of their compositions.

1 Motivation

It is often very useful and inspirative to see the same thing from more different points of view. This general sentence we can applied to Formal Concept Analysis. Guo-Quiang Zhang in this paper [12] have considered a concept lattice in the terms of the category theory. As he says, his paper brings these (originally independent) areas together and establishes fundamental connections among them, leaving open opportunities for the exploration of cross-disciplinary influences. He emphasizes the substantial culture differences among these fields: Formal Concept Analysis focuses on internal properties of and algorithms for concept structures almost exclusively on an individual basis, while the Category Theory mandates that concept structures should be looked at collectively as a whole with appropriate morphisms relating one individual structure to another. (Note that Zhang speaks about the third area, Domain Theory, too, but we will not focus on this part of his considerations.)

We will continue in this direction of research. In the papers [7] and [8] we define a new type of fuzzification of Formal Concept Analysis, a so-called generalized concept lattice, which moreover in some sense generalizes some other fuzzy constructions of concept lattices (namely a fuzzy concept lattice, an one-sided concept lattice and a concept lattice with hedges, all mentioned below). A natural questions arise: If a notion of Chu space is a pendant of a classical (crisp, Ganter & Wille's) concept lattice in the category theory, what object

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will be a categorial counterpart of this generalized concept lattice? And what about morphisms of such objects? We will try to answer to the first question and partially answer to the second one in this paper.

Proofs of all lemmas are omitted because of lack of space.

2 Concept lattices and Chu spaces

By a Chu space ([12]) we understand a triple (A, B, R), where A is a (non-empty) set of attributes, B is a (non-empty) set of objects, and R is a subset of $A \times B$.

It is easy to see that a notion of Chu space corresponds to a notion of context leading to a classical concept lattice ([6]): For a Chu space (A, B, R) define the following mappings $\uparrow : \mathfrak{P}(B) \to \mathfrak{P}(A)$ and $\downarrow : \mathfrak{P}(A) \to \mathfrak{P}(B)$:

If $X \subseteq B$ and $Y \subseteq A$ then

$$X^{\uparrow} = \{a \in A : (\forall b \in X)\langle a, b\rangle \in R\},$$

$$Y^{\downarrow} = \{b \in B : (\forall a \in Y)\langle a, b\rangle \in R\}.$$

These two mappings form *Galois connection*. By *concept* we understand a pair $\langle X, Y\rangle$ such that $X^{\uparrow} = Y$ and $Y^{\downarrow} = X$. The set of all concepts is called a *concept lattice*. It is proven (in the basic book [6]) that it is really a lattice (moreover complete).

Theorems on one single Chu space / concept lattice are rather static, they describe status quo of it. But it is often inspirative to see things dynamically. In this case we can want to ask how concepts will changed if a new object is added to context. The following notion brings dynamics to such considerations:

By a Chu mapping from a Chu space (A_1, B_1, R_1) to a Chu space (A_2, B_2, R_2) we will understand a pair of functions $\langle p, q\rangle$ with $p : A_2 \to A_1$ and $q : B_1 \to B_2$ (note that indices are interchanged in p) satisfying $\langle p(a_2), b_1\rangle \in R_2$ iff $\langle a_2, q(b_1)\rangle \in R_1$ for all $a_2 \in A_2$ and $b_1 \in B_1$.

In this framework an adding of new row to a table means the change from one Chu space to another. This construction induces a Chu mapping $\langle p, q\rangle$ from the initial Chu space to the enlarged one such that p and q are identities.

It is easy to see that all Chu spaces and Chu mappings as their morphisms form a category.

3 *L*-liftings

In the next sections we will need this L-fuzzification of the image and the inverse image of a set:

Let L will be (the support of) a complete lattice, S and T be arbitrary sets and $h : S \to T$. Then define the *canonical L-liftings* $h_L^+ : L^S \to L^T$ (*forward L-image*) and $h_L^- : L^T \to L^S$ (*inverse L-image*) in the following way:

– If $g : S \to L$ then $h_L^+(g) : T \to L$ is defined by the formula

$$h_L^+(g)(t) = \sup\{g(s) : h(s) = t\}.$$

– If $f : T \to L$ then $h_L^-(f) : S \to L$ is defined by the formula

$$h_L^-(f)(s) = (h \circ f)(s) = f(h(s))$$

(i.e. $h_L^-(f) = h \circ f$).

In the special case $L = \{0,1\}$ we really obtain the coincidence between $h[X]$ and $h_L^+(\chi_X)$ (or loosely $h_L^+(X) \approx h[X]$) and between $h^{-1}[Y]$ and $h_L^-(\chi_Y)$ (or loosely $h_L^-(Y) \approx h^{-1}[Y]$), namely:

Lemma 1.

$$h^+_{\{0,1\}}(\chi_X) = \chi_{h[X]}$$

for arbitrary $X \subseteq S$, i.e. $\chi_X : S \to \{0,1\}$.

Lemma 2.

$$h^-_{\{0,1\}}(\chi_Y) = \chi_{h^{-1}[Y]}$$

for arbitrary $Y \subseteq T$, i.e. $\chi_Y : T \to \{0,1\}$.

The compositions of our liftings $h_L^+ \circ h_L^- : L^S \to L^S$ and $h_L^- \circ h_L^+ : L^T \to L^T$ fulfills these interesting properties:

Lemma 3. *a1) $(h_L^+ \circ h_L^-)(g) \geq g$ (pointwise) for all $g : S \to L$.*
a2) $h_L^+ \circ h_L^-$ is the identity on L^S iff h is an injection.
a3) $(h_L^+ \circ h_L^-)(g) = (h_L^+ \circ h_L^-) \circ (h_L^+ \circ h_L^-)(g)$ for all $g : S \to L$.
a4) If $g_1 \leq g_2$ (pointwise) then $(h_L^+ \circ h_L^-)(g_1) \leq (h_L^+ \circ h_L^-)(g_2)$ for all $g_1, g_2 : S \to L$.
b1) $(h_L^- \circ h_L^+)(f) \leq f$ for all $f : T \to L$.
b2) $h_L^- \circ h_L^+$ is the identity on L^T iff h is a surjection.
b3) $(h_L^- \circ h_L^+)(g) = (h_L^- \circ h_L^+) \circ (h_L^- \circ h_L^+)(f)$ for all $f : T \to L$.
b4) If $f_1 \leq f_2$ (pointwise) then $(h_L^- \circ h_L^+)(f_1) \leq (h_L^- \circ h_L^+)(f_2)$ for all $f_1, f_2 : T \to L$.

We can summarize properties 1), 3) and 4) (a kernel operator is the dual notion to a closure operator):

Corrolary 1 *a) $h_L^+ \circ h_L^-$ is a closure operator.*
b) $h_L^- \circ h_L^+$ is a kernel operator.

4 A generalized concept lattice

An idea of defining of a generalized concept lattice arose as an answer to the natural question of looking for a common platform for so far known fuzzifications of a classical crisp concept lattice.

Let us recall its definition and basic properties ([7]):

Let P be a poset, C and D be complete lattices. Let $\bullet : C \times D \to P$ be isotone and left-continuous in both their arguments. Let A and B be non-empty sets and let R be P-fuzzy relation on their Cartesian product, i.e. $R : A \times B \to P$.

Define the following mapping $\uparrow: D^B \to C^A$:
If $g : B \to D$ then $\uparrow(g) : A \to C$ is defined in the following way:

$$\uparrow(g)(a) = \sup\{c \in C : (\forall b \in B) c \bullet g(b) \leq R(a, b)\}.$$

Symmetrically we define the mapping $\downarrow: C^A \to D^B$:
If $f : A \to C$ then $\downarrow(f) : B \to D$ is defined in the following way:

$$\downarrow(f)(b) = \sup\{d \in D : (\forall a \in A) f(a) \bullet d \leq R(a, b)\}.$$

Mappings $\downarrow$ and $\uparrow$ form a Galois connection, i.e.

1a) $g_1 \leq g_2$ implies $\uparrow(g_1) \geq \uparrow(g_2)$.
1b) $f_1 \leq f_2$ implies $\downarrow(f_1) \geq \downarrow(f_2)$.
2a) $g \leq \downarrow(\uparrow(g))$.
2b) $f \leq \uparrow(\downarrow(f))$.

Then the pair of functions $\langle g, f \rangle$ from $D^B \times C^A$ such that $g^\uparrow = f$ and $f^\downarrow = g$, is called a *(generalized) concept.* If $\langle g_1, f_1 \rangle$ and $\langle g_2, f_2 \rangle$ are concepts, we will write $\langle g_1, f_1 \rangle \leq \langle g_2, f_2 \rangle$ iff $g_1 \leq g_2$ (or equivalently $f_1 \geq f_2$). The set of all such concepts with the order $\leq$ is called a *(generalized) concept lattice* and denoted by $\mathrm{GCL}(A, B, R, C, D, P, \bullet)$. The appropriate analogy of the Basic Theorem on Concept Lattice can be formulated about it, it is proven in [7] and [8].

This construction is really a generalization of known fuzzifications of concept lattice. For Pollandt's ([11]) and Bělohlávek's ([1], [2]) fuzzy concept lattice we have $C = D = P = L$ and $\bullet$ is the product, for an one-sided fuzzy concept lattice ([5], [10], [3]) $C = P = [0, 1]$ but $D = \{0, 1\}$ and $\bullet$ is the minimum (or the product again).

Note that this notion is not the only common platform for notions of an one-sided fuzzy concept lattice and of a fuzzy concept lattice. The alternative answer is the approach given by Bělohlávek et al. ([4]) again. They define a so-called concept lattice with hedges. In the paper [9] we show that this construction can be understood as a special case of a generalized concept lattice.

5 A generalized Chu space

Now we will try to express these ideas by means of the category theory. We define a notion of generalized Chu space what will be a pendant of a generalized concept lattice.

Let A and B be non-empty sets, P be (the support of) a poset, C and D be (the supports of) complete lattices and $\bullet : C \times D \to P$ be isotone and left-continuous in both its arguments. By a *generalized Chu space* it will be understood the tuple $(A, B, R, P, C, D, \bullet)$.

It is easy to see that an ordinary Chu space can be seen as a special case of a generalized Chu space (A, B, R') for $P = \{0, 1\}$ where R is the characteristic function of the relation R', i.e.

$$R(a,b) = \begin{cases} 1, & \text{if } \langle a,b\rangle \in R', \\ 0, & \text{if } \langle a,b\rangle \notin R', \end{cases}$$

$C = D = \{0,1\}$ (a set is identified with its characteristic function) and $\bullet$ is the product.

For a generalized Chu space $(A,B,R,P,C,D,\bullet)$ we can again define functions $\uparrow\colon D^B \to C^A$ and $\downarrow\colon C^A \to D^B$ in the same way as before. Hence it is easy to see that the generalized Chu space $(A,B,R,P,C,D,\bullet)$ naturally leads to the corresponding generalized concept lattice $\mathrm{GCL}(A,B,R,P,C,D,\bullet)$.

If we want to speak about some category, we should say something about its morphisms. In our case of generalized Chu space which will be objects of our category we can imagine more types of such morphisms. For simplicity (as a beginning of this approach) we will focus to morphisms which work only with a fixed C, D, P and $\bullet$ of generalized Chu spaces $(A,B,R,P,C,D,\bullet)$:

Let $(A_1,B_1,R_1,P,C,D,\bullet)$ and $(A_2,B_2,R_2,P,C,D,\bullet)$ be generalized Chu spaces (note that C, D, P and $\bullet$ are the same for both). Let $p : A_2 \to A_1$ and $q : B_1 \to B_2$ (note the mutually inverse directions again) are such that

$$R_2(a_2, q(b_1)) = R_1(p(a_2), b_1)$$

holds for all $a_2 \in A_2$ and $b_1 \in B_1$. Then $\langle p,q\rangle$ will be called an *object-attribute (OA-) morphism* of these generalized Chu spaces.

It is easy to see the following:

Lemma 4. *The system of all generalized Chu spaces and their OA-morphisms is a category.*

6 Two commutative diagrams

In this section we assume that $(A_1,B_1,R_1,P,C,D,\bullet)$ and $(A_2,B_2,R_2,P,C,D,\bullet)$ are generalized Chu spaces, $\langle p,q\rangle$ is some their object-attribute (OA-) morphism, and the mappings $\downarrow_i$ and $\uparrow_i$ correspond to the space $(A_i,B_i,R_i,P,C,D,\bullet)$ and they are defined as before. We can see the following properties:

Lemma 5. *a) If p and q are surjective then $\uparrow_2 \circ p_C^+ = q_D^- \circ \uparrow_1$.*
b) $\uparrow_1 \circ p_C^- = q_D^+ \circ \uparrow_2$.

$$\begin{array}{ccc} D^{B_2} & \xrightarrow{\uparrow_2} & C^{A_2} \\ \big\downarrow q_D^- & \# & \big\downarrow p_C^+ \\ D^{B_1} & \xrightarrow{\uparrow_1} & C^{A_1} \end{array} \qquad\qquad \begin{array}{ccc} D^{B_1} & \xrightarrow{\uparrow_1} & C^{A_1} \\ \big\downarrow q_D^+ & \# & \big\downarrow p_C^- \\ D^{B_2} & \xrightarrow{\uparrow_2} & C^{A_2} \end{array}$$

7 Conclusions and future work

In this paper we try to continue in ideas of the work of Zhang how to look at a concept lattice from the categorial point of view. We define very naturally a appropriate modification of Chu spaces to our generalized concept lattice. It seems that there are more ways to define morphisms between such generalized Chu spaces, but for the beginning we started with mappings which transform the sets of objects and attributes only. Moreover we have defined a fuzzy version of liftings of a function and discuss their basic properties and some commutative relationships to the mappings defining a generalized concept lattice.

We have say that a generalized concept lattice is a generalization of till known fuzzifications of concept lattice. But what does mean the word generalization here? It is rather intuitive, because all these constructions lead to (all) complete lattices. We hope that the more precise answer will be given by the category theory, maybe it will be the existence of some canonical mapping, i.e. a functor between corresponding Chu spaces.

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