On Concept Lattices of Efficiently Solvable Voting Protocols

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Voting protocols are solvable with respect to some game-theoretic solution concept or rule if for any admissible preference profile there exists a non-empty set of solutions, and efficiently solvable if the resulting outcomes are Pareto efficient. Thus efficiently solvable protocols are of special interest in that they both enjoy a suitably defined strategic robustness and ensure Pareto-efficiency of the resulting strategically stable outcomes.

The present paper is devoted to the introduction and study of the concept lattices of some prominent classes of efficiently solvable voting protocols. That study is meant to provide some insight on the 'structural' distribution of decision power among coalitions induced by a few protocols that are 'nicely robust' with respect to some prominent game-theoretic solution concepts.

Model. Let $N = \{1, .., n\}$ denote the set of *players*, $X = \{x_1, .., x_k\}$ the set of *outcomes*, and $\mathbf{R}_X$ the set of linear (preference) orders on X. A strategic game form on (N, X) is an array $\Gamma = (N, X, (S_i)_{i \in N}, h)$ where S_i is a set, the *strategy set* of player $i, i = 1, .., n$ and $h : \prod_{i=1}^{n} S_i \rightarrow X$ is a surjective function, the *outcome function* of Γ.A *voting protocol* $(N, X, \mathbf{R}_X)$ is a strategic game form $\Gamma = (N, X, (S_i)_{i \in N}, h)$ such that $S_i \supseteq \mathbf{R}_X$ for some $i \in N$.

A *solution concept* is a rule for solving games of a certain collection: e.g. if $\mathbf{R}_X$ is the set of admissible preferences for each player $i \in N$, then a solution concept for the set $\Gamma(\mathbf{R}_X^N) = \{(\Gamma, R^N) : R^N \in \mathbf{R}_X^N\}$ of games induced by game form Γ on the domain $\mathbf{R}_X^N$ of all profiles of total preference preorders on X is a correspondence $\sigma : \Gamma(\mathbf{R}_X^N) \rightarrow\rightarrow \prod_{i=1}^{n} S_i$. A few concrete examples of solution concepts including Nash equilibrium, strong equilibrium and coalitional equilibrium with threats will be introduced and discussed below.

Let Γ be a voting protocol and σ a solution concept for $\Gamma(\mathbf{R}_X^N)$.Then, voting protocol Γ is said to be *σ-solvable* over preference domain $\mathbf{R}_X^N$ if $\sigma((\Gamma, R^N)) \neq \emptyset$ for any $R^N \in \mathbf{R}_X^N$.

Moreover, at any profile of preference preorders $R^N = (R_1, .., R_n) \in \mathbf{R}_X^N$ we denote $Par(R^N)$ the set of *Pareto efficient outcomes.* Voting protocol Γ

will be said to be *efficiently σ-solvable* over preference domain $\mathbf{R}_X^N$ if $\emptyset \neq h[\sigma((\Gamma, R^N))] \subseteq Par(R^N)$ for any $R^N \in \mathbf{R}_X^N$.A *coalitional game form* is a triple $\mathbf{G} = (N, X, E)$ where N and X are non-empty sets denoting the sets of players and outcomes, respectively, and $E : \mathcal{P}(N) \to \mathcal{P}(\mathcal{P}(X))$ is the coalitional power function: the 'power-value' $E(S)$ of coalition $S \subseteq N$ is the collection of all events $A \subseteq X$ coalition $S \subseteq N$ is able to 'force' (under some suitable interpretation of the latter notion).We are mainly interested in those CGF that can represent the decision power of coalitions under a voting protocol Γ. In particular, we look at the $\alpha - CGF$ of Γ- denoted by $E_\alpha(\Gamma)$- as defined by the following rule which describes for each coalition the set of events in $\mathcal{P}(X)$ that coalition is able to force *sticking to a certain constant strategy profile.* Now, the *concept lattice* $\mathbf{L}(\Gamma)$ of Γ is the concept lattice of $E_\alpha(\Gamma)$ when regarded as a binary relation between $\mathcal{P}(N)$ and $\mathcal{P}(X)$.

Results. A strategy of player i in game g is *dominant* if no matter what the other choose its outcomes are never worse and occasionally better.Let us denote $DS_i(g)$ the set of dominant strategies for i, and $DS(g) = \Pi_i DS_i(g)$ the set of all profiles of dominant strategies of game g. Then, a voting protocol $\Gamma = (N, X, (S_i)_{i\in N}, h)$ is dominant-strategy solvable (or DS-solvable) over preference domain $\mathbf{R}_X^N$ if $DS(g) \neq \emptyset$ for any game $g = (\Gamma, R^N)$ with $R^N \in \mathbf{R}_X^N$.

Proposition 1 *Let $\Gamma = (N, X, (S_i)_{i\in N}, h)$ be an efficiently DS-solvable voting protocol, $\mathbf{G}_\alpha(\Gamma) = (N, X, E_\alpha(\Gamma)), \mathbf{G}_\beta(\Gamma) = (N, X, E_\beta(\Gamma))$. Then $\mathbb{C}(\mathbf{G}_\alpha(\Gamma)) = \mathbb{C}(\mathbf{G}_\beta(\Gamma)) \simeq \mathbf{4}$.*

Notice that the concepts attached to a DS-solvable voting protocol may be described by the following intents: 'omnipotent' coalitions (i.e. coalitions able to enforce any event including the null event), 'all-powerful' coalitions (i.e. coalitions able to enforce any event *except for* the null event), 'essentially powerless' coalitions, and 'absolutely powerless' coalitions. The corresponding extents are, respectively, the empty set, the ultrafilter of all coalitions including the *dictator* among their members, the set of all nonempty coalitions, and the set of all coalitions.

Next, we consider efficiently Nash equilibrium solvable voting protocols (such protocols are usually referred to as *acceptable*). Let $g = (N, X, (S_i)_{i\in N}, h, (R_i)_{i\in N})$ be a game in strategic form.The set of Nash equilibria of game g is denoted by $NE(g)$. One kind of 'universal' family of efficiently Nash-equilibrium-solvable voting protocols $\Gamma^{M(.)} = (N, X, (S_i^{M(.)})_{i\in N}, h^{M(.)})$ -the family of so-called *Maskin protocols* (see again Danilov, Sotskov(2002)) - is known to exist when $|N| \geqslant 3$.

Proposition 2 *Let $f : \mathbf{R}_X^N \to\to X$ be a Maskin-monotonic and Pareto efficient non-empty valued correspondence and $\Gamma^{M(f)} = (N, X, (S_i^{M(f)})_{i\in N}, h^{M(f)})$ the corresponding Maskin voting protocol. Then,*

$\mathbb{C}(\mathbf{G}_\alpha(\Gamma^{M(f)})) = \mathbb{C}(\mathbf{G}_\beta(\Gamma^{M(f)})) \simeq \mathbf{4}$.

The concepts attached to the Maskin voting protocol may also be described by the following intents: 'omnipotent' coalitions (i.e. coalitions able to enforce any event including the null event), 'all-powerful' coalitions (i.e. coalitions able to enforce any event *except for* the null event), 'essentially powerless' coalitions, and 'absolutely powerless' coalitions. The corresponding extents, are respectively the empty set, *the set of all coalitions including at least* $n-1$ agents, the set of all nonempty coalitions, and the set of all coalitions.

Finally, we turn to some solution concepts implying coalition formation and efficient coordinated coalitional behaviour. Again, let $g = (N, X, (S_i)_{i \in N}, h, (R_i)_{i \in N})$ be a game in strategic form. A *strong equilibrium* of g is a strategy profile $s = (s_i)_{i \in N}$ such that for any coalition $T \subseteq N$ and any $s'_T \in \prod_{i \in T} S_i$ there exists $i \in T$ such that $h(s_T, s_{N \setminus T}) R_i h(s'_T, s_{N \setminus T})$. A *coalitional equilibrium with threats of* g is a strategy profile $s = (s_i)_{i \in N}$ such that for any coalition $T \subseteq N$ and any $s'_T \in \prod_{i \in T} S_i$ there exists $s'_{N \setminus T} \in \prod_{i \in N \setminus T} S_i$ and $i \in T$ such that $h(s_T, s_{N \setminus T}) R_i h(s'_T, s'_{N \setminus T})$ (notice that coalitional equilibrium *outcomes* with threats of g do exactly coincide with the *core* outcomes of g). It turns out that certain *voting-by-limited-veto* protocols enjoy both strong equilibrium solvability and coalitional-equilibrium-with-threats solvability over preference domain $\mathbf{R}_X^N$. It can be shown that the following proposition holds true

Proposition 3 *Let* $\mathbf{G}^{PV} = (N, X, E^{PV})$ *be the proportional veto EF (where each coalition can veto a number of outcomes essentially proportional to its size. Then,* $\mathbf{L}(\mathbf{G}^{PV}) = \mathbf{1} \oplus \mathbf{n} \oplus \mathbf{1}$ *(where* $\mathbf{n}$ *denotes the chain of size* n *).*

Clearly, the intents of the concepts attached to the proportional veto protocol may be described as 'the coalitions that are able to veto at least $k \cdot l$ outcomes', with $k \cdot l \leqslant k \cdot n$. The corresponding extents amount to player (sub)sets of cardinality l, $0 \leqslant l \leqslant n$.It should also be remarked that the concept lattices of non-anonymous versions of that voting by veto protocol are also chains. It remains to be seen whether efficiently solvable voting protocols endowed with more general concept lattices do exist.

References

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