# Generalizations of Approximable Concept Lattice

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**Abstract.** B. Ganter, R. Wille initiated formal concept analysis, concept lattice is one of the main notions and tools, see [12]. Some researchers have investigated the fuzzification of the classical crisp concept lattice. In [1], from the point of view of fuzzy logic, R. Bělohlávek investigated concept lattice in fuzzy setting. In [16, 17], S. Krajči studied generalized concept lattice.
On the other hand, as a generalization of concept, in [15, 21, 22], Zhang, P. Hitzler, Shen defined the notion of approximable concept on a Chu space. In this paper, we introduce two generalizations of approximable concept lattice: approximable concept lattice in the sense of R. Bělohlávek, and generalized approximable concept in the sense of S. Krajči.

**Keywords:** concept, approximable concept, L-set, generalized concept, Chu space

## 1 Introduction

B. Ganter, R. Wille initiated formal concept analysis, which is an order-theoretical analysis of scientific data. Concept lattice is one of the main notions and tools, see [12]. Some researchers have investigated the fuzzification of the classical crisp concept lattice. One is R. Bělohlávek's work ([1]), which considers (L-)fuzzy subsets of objects and (L-)fuzzy subsets of attributes. Another is S. Krajči's work ([18]) which considers fuzzy subsets of attributes and ordinary/classical/crisp subsets of objects. For more details, see [1, 5, 6, 7, 16, 17, 18].

As constructive models of linear logic, Barr and Seely brought Chu space to light in computer science. V. Pratt also investigated Chu space in [19, etc.], and Zhang, P. Hitzler, Shen discussed a special form of Chu space in [15, 21, 22].

From the study of domain theory, Zhang showed that a concept is not an affirmable property([20]), see Example 2. As a generalization of concept, in [15, 21, 22], Zhang, P. Hitzler, Shen introduced the notion of approximable concept on a Chu space. They obtained the equivalence between the category of formal contexts with context morphisms, and the category of complete algebraic lattices with Scott continuous functions. For more results, including its applications in data-mining and knowledge discovery, we refer to [21, 22].

In [8], we investigated the relation between approximable concept lattice and formal topology (information base). Thus the connections between the four categories of Domain theory, Formal Concept Analysis, Formal topology, Information System have been constructed in [8, 21, 22].

In this paper, we begin with an overview of algebraic lattices, **L**-sets, which surveys Preliminaries. In Section 3, we introduce Zhang's work. Then in Section 4, we discuss the equivalence between two definitions of approximable concept in fuzzy setting. In the end, we investigate generalized approximable concept, and show that generalized approximable concept lattices represent algebraic lattices.

## 2 Preliminaries

Let us recall some main notions needed in the paper. i.e., algebraic lattices, **L**-sets. For the other notions, see [3, 13].

### 2.1 Algebraic Lattices

In [13], the notions of continuous lattice and algebraic lattice were introduced. In the section, we recall some main definitions. For more details, see [13].

Let $(P, \leq, \vee, \wedge, 0, 1)$ be a complete lattice. For $D \subseteq P$, $D$ is called a directed set, $\forall x, y \in D$, if there exists $z \in D$, such that $x \leq z, y \leq z$.

For $x, y \in P$, $x$ is said to be way below $y$, denoted by $x \ll y$, if for all directed set $D$ with $y \leq \vee D$, there exists $z \in D$, such that $x \leq z$. Let $\Downarrow x = \{y \mid y \ll x\}$. $(P, \leq)$ is called a continuous lattice if for every $x \in P$, we have $x = \vee \Downarrow x$.

$x \in P$ is called a compact element, if $x \ll x$, which is equivalent to: for all directed sets $D$ with $x \leq \vee D$, there exists $z \in D$, satisfying $x \leq z$. Let $K(\ll) = \{x \mid x \text{ is compact }\}$, $K(\ll)$ is not a complete lattice in general.

$(P, \leq)$ is called an algebraic lattice, if for every $x \in P$, there exists a directed set $D_x$ of compact elements, such that $x = \vee D_x$, that is to say,

$x = \vee(\downarrow x \cap K(\ll))$,

where $\downarrow x = \{y \mid y \leq x\}$.

In universal algebra, algebraic lattices have become familiar objects as lattices of congruences and lattices of subalgebras of an algebra. Thus they have been extensively studied, and applied in many areas, such that topological theory and domain theory (see [11]). The role of algebraic completely lattice **L**-ordered sets is analogous to the role of algebraic lattices in ordinary relational systems.

### 2.2 L-Sets

The notion of an L-set was introduced in ([14]), as a generalization of Zadeh's (classical) notion of a fuzzy set. An overview of the theory of L-sets and L-relations (i.e., fuzzy sets and relations in the framework of complete residuated lattices) can be found in [3]. Let us recall some main definitions.

**Definition 1.** A residuated lattice is an algebra $\mathbf{L}=\langle L,\vee,\wedge,\otimes,\rightarrow,0,1\rangle$ such that

(1) $\langle L,\vee,\wedge,\otimes,\rightarrow,0,1\rangle$ is a lattice with the least element 0 and the greatest element 1.

(2) $\langle L,\otimes,1\rangle$ is a commutative monoid, i.e., $\otimes$ is associative, commutative, and it holds the identity $a\otimes 1=a$.

(3) $\otimes,\rightarrow$ form an adjoint pair, i.e.,

$x\otimes y\leq z$ iff $x\leq y\rightarrow z$ holds for all $x,y,z\in L$.

Residuated lattice $\mathbf{L}$ is called complete if $\langle L,\vee,\wedge\rangle$ is a complete lattice. In this paper, we assume that $\mathbf{L}$ is complete.

The following properties of complete residuated lattices maybe needed in this paper.

(1) $a\leq b\Rightarrow a\rightarrow c\geq b\rightarrow c$, (2) $a\leq b\Rightarrow c\rightarrow a\leq c\rightarrow b$,

(3) $a\leq b\Rightarrow a\otimes c\leq b\otimes c$, (4) $a=1\rightarrow a$,

(5) $a\otimes b\leq a\wedge b$, (6) $a\leq (a\rightarrow b)\rightarrow b$,

(7) $a\otimes (a\rightarrow b)\leq b$, (8) $a\otimes (b\rightarrow c)\leq b\rightarrow a\otimes c$,

(9) $a\otimes \bigwedge_{i\in I} b_i\leq \bigwedge_{i\in I}(a\otimes b_i)$, (10) $(\bigvee_{i\in I} a_i)\rightarrow b=\bigwedge_{i\in I}(a_i\rightarrow b)$,

(11) $a\rightarrow \bigwedge_{i\in I} b_i=\bigwedge_{i\in I}(a\rightarrow b_i)$, (12) $(a\rightarrow b)\otimes(b\rightarrow c)\leq (a\rightarrow c)$.

As discussed in [3], several important algebras are special residuated lattices: Boolean algebras, Heyting algebras, BL-algebras, MV-algebras, Girard monoids and others.

For a universe set $X$, an $\mathbf{L}$-set in $X$ is a mapping $A:X\rightarrow L$, $A(x)$ indicates that the truth degree of "$x$ belongs to $A$". We use the symbol $L^X$ to denote the set of all $\mathbf{L}$-sets in $X$. The concept of an $\mathbf{L}$-relation is defined obviously, and the truth degree to which elements $x$ and $y$ are related by an $\mathbf{L}$-relation $I$ is denoted by $I(x,y)$ or $(xIy)$.

For $a\in L, x\in X$, $\{a/x\}$ is defined as an $\mathbf{L}$-set in $X$: $\{a/x\}(x)=a$, $\{a/x\}(y)=0$, if $y\neq x$.

A binary $\mathbf{L}$-relation $\approx$ on $X$ is an $\mathbf{L}$-equality if it satisfies: $\forall x,y,z\in X$, $(x\approx x)=1$(reflexivity), $(x\approx y)=(y\approx x)$ (symmetry), $(x\approx y)\otimes(y\approx z)\leq (x\approx z)$(transitivity), and $(x\approx y)=1$ implies $x=y$.

$I\in L^{X\times Y}$ is a binary $\mathbf{L}$-relation, and it is compatible with respect to $\approx_X$ and $\approx_Y$ if $I(x_1,y_1)\otimes(x_1\approx_X x_2)\otimes(y_1\approx_Y y_2)\leq I(x_2,y_2)$ for any $x_i\in X, y_i\in Y (i=1,2)$. Analogously, $A\in L^X$ is compatible with respect to $\approx_X$ if $A(x_1)\otimes(x_1\approx_X x_2)\leq A(x_2)$. An $\mathbf{L}$-set $A\in L^{\langle X,\approx\rangle}$ is called an $\approx$-singleton if there exists $x_0\in X$, such that $A(x)=(x\approx x_0)$ for any $x\in X$.

An $\mathbf{L}$-order on $X$ with an $\mathbf{L}$-equality relation $\approx$ is a binary $\mathbf{L}$-relation $\preceq$ which is compatible with respect to $\approx$ and satisfies: $\forall x,y,z\in X$ $(x\preceq x)=1$(reflexivity), $(x\preceq y)\wedge(y\preceq x)\leq (x\approx y)$ (antisymmetry), $(x\preceq y)\otimes(y\preceq z)\leq (x\preceq z)$ (transitivity). A set $X$ equipped with an $\mathbf{L}$-order $\preceq$ and an $\mathbf{L}$-equality $\approx$ is called an $\mathbf{L}$-ordered set $\langle\langle X,\approx\rangle,\preceq\rangle$.

These notions are generalizations of the classical notions. Indeed, if $\mathbf{L}=\mathbf{2}$, $\mathbf{L}$-order $\preceq$, $\mathbf{L}$-equality $\approx$ coincide with the classical order $\leq$ and equality $=$.

For $A, B \in L^X$, we define $S(A,B) = \bigwedge_{x\in X} A(x) \to B(x)$, $(A \approx B) = \bigwedge_{x\in X} A(x) \leftrightarrow B(x)$, and $(A \preceq B) = S(A,B)$, thus $\langle\langle L^X, \approx\rangle, \preceq\rangle$ is an **L**-ordered set, see Example 1. We write $A \subseteq B$, if $S(A,B) = 1$.

**Example 1.** This is [1] Example 6(1). For $\emptyset \neq M \subseteq L^X$, we obtain that $\langle\langle M, \approx\rangle, S\rangle$ is an **L**-ordered set. In fact, reflexivity and antisymmetry are trivial, we have to prove transitivity and compatibility. Transitivity: $S(A,B) \otimes S(B,C) \leq S(A,C)$ holds if and only if $S(A,B) \otimes S(B,C) \leq A(x) \to C(x)$, i.e., $\forall x \in X$, $A(x) \otimes S(A,B) \otimes S(B,C) \leq C(x)$, and it is true since $A(x) \otimes S(A,B) \otimes S(B,C) \leq A(x) \otimes (A(x) \to B(x)) \otimes (B(x) \to C(x)) \leq C(x)$. In the similarly way, we also prove Compatibility: $S(A,B) \otimes (A \approx A') \otimes (B \approx B') \leq S(A', B')$.

For $S(A,B)$, Lemma 1 will be used in the paper, see [3].

**Lemma 1.** (1) $S(A, \bigcap_{i\in I} B_i) = \bigwedge_{i\in I} S(A, B_i)$,

(2) $A(x) \otimes S(A,B) \leq B(x)$.

Suppose $X$ and $Y$ are two sets with **L**-equalities $\approx_X$ and $\approx_Y$, respectively. An **L**-Galois connection ([1]) between $\langle X, \approx_X\rangle$ and $\langle Y, \approx_Y\rangle$ is a pair $\langle {}^{\uparrow}, {}^{\downarrow}\rangle$ of mappings ${}^{\uparrow} : L^{\langle X,\approx_X\rangle} \to L^{\langle Y,\approx_Y\rangle}$, ${}^{\downarrow} : L^{\langle Y,\approx_Y\rangle} \to L^{\langle X,\approx_X\rangle}$, and satisfying the following conditions:

$S(A_1, A_2) \leq S(A_2^{\uparrow}, A_1^{\uparrow})$, $S(B_1, B_2) \leq S(B_2^{\downarrow}, B_1^{\downarrow})$,

$A \subseteq A^{\uparrow\downarrow}$, and $B \subseteq B^{\downarrow\uparrow}$ for any $A, A_1, A_2 \in L^X, B, B_1, B_2 \in L^Y$.

A mapping $C : L^X \to L^Y$ is an **L**-closure operator, if for $A, B \in L^X$, we have (1) $A \subseteq C(A)$, (2) $S(A,B) \leq S(C(A), C(B))$, and (3) $C(C(A)) = C(A)$.

## 3 Approximable Concepts introduced by Zhang

As showed in Introduction, Zhang, P. Hitzler, Shen considered a special form of Chu space in [15, 21, 22] as follows.

**Definition 2**. A Chu space $P$ is a triple $P = (P_o, \models_P, P_a)$, where $P_o$ is a set of objects and $P_a$ is a set of attributes. The satisfaction relation $\models_P$ is a subset of $P_o \times P_a$. A mapping from a Chu space $P = (P_o, \models_P, P_a)$ to a Chu space $Q = (Q_o, \models_Q, Q_a)$ is a pair of functions $(f_a, f_o)$ with $f_a : P_a \to Q_a$ and $f_o : Q_o \to P_o$ such that for any $x \in P_a$ and $y \in Q_o$, $f_o(y) \models_P x$ iff $y \models_Q f_a(x)$.

With respect to a Chu space $P = (P_o, \models_P, P_a)$, two functions can be defined:

$\alpha : \mathcal{P}(P_o) \to \mathcal{P}(P_a)$ with $X \to \{a \mid \forall x \in X\ x \models_P a\}$,

$\omega : \mathcal{P}(P_a) \to \mathcal{P}(P_o)$ with $Y \to \{o \mid \forall y \in Y\ o \models_P y\}$.

$\alpha, \omega$ form a pair of Galois connection between $\mathcal{P}(P_o)$ and $\mathcal{P}(P_a)$ ([13]).

A subset $A \subseteq P_a$ is called an intent of a formal concept if it is a fixed point of $\alpha \circ \omega$, i.e., $\alpha(\omega(A)) = A$, in [21], $A$ is also called a (formal) concept (of attributes).

If $A$ is a concept, for every subset $B \subseteq A$, we have $B \subseteq \alpha(\omega(B)) \subseteq \alpha(\omega(A)) = A$ ([22]).

Dually, an extent of a formal concept, or a (formal) concept (of objects) also defined in [22].

Zhang pointed out in [22] that in FCA, a Chu space is called a formal context, but "Chu" carries with it the notion of morphism, to form a category. On the other hand, FCA provides the notion of concept, intrinsic to a Chu space.

As a generalization of the notion of concept, Zhang and Shen introduced the notion of approximable concept in [21], a subset $A \subseteq P_a$ is an approximable concept (of attributes) if for every finite subset $X \subseteq A$, we have $\alpha(\omega(X)) \subseteq A$. Clearly, every concept is an approximable concept, but the converse is false ( see Example 2).

**Example 2** In [21], Zhang and Shen gave the following example, to show an approximable concept is not a concept in general.

| P | $\uparrow t$ | $\uparrow b$ | $\uparrow *$ | $\uparrow 0$ | $\uparrow 1$ | $\uparrow 2$ | $\cdots$ | $\uparrow -1$ | $\uparrow -2$ | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|
| t | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\cdots$ | $\times$ | $\times$ | $\cdots$ |
| b | | $\times$ | | | | | $\cdots$ | | | $\cdots$ |
| $*$ | | $\times$ | $\times$ | | | | $\cdots$ | | | $\cdots$ |
| 0 | | $\times$ | | $\times$ | | | $\cdots$ | $\times$ | $\times$ | $\cdots$ |
| 1 | | $\times$ | | $\times$ | $\times$ | | $\cdots$ | $\times$ | $\times$ | $\cdots$ |
| $\vdots$ | | $\vdots$ | | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| -1 | | $\times$ | | | | | $\cdots$ | $\times$ | $\times$ | $\cdots$ |
| -2 | | $\times$ | | | | | $\cdots$ | | $\times$ | $\cdots$ |
| $\vdots$ | | $\vdots$ | | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

Given $S = \{b, \cdots, -2, -1, 0, 1, 2, \cdots, t, *\}$, with the order $b < \cdots < -2 < -1 < 0 < 1 < 2 < \cdots < t$, and $b < * < t$. Then $S$ is a complete lattice which is not algebraic.

$\forall x \in S$, let $\uparrow x = \{y \mid x \leq y\}$, $\times$ indicates that $y \in \uparrow x$, we obtain a Chu space as follows.

$P_a = \{\uparrow x \mid x \in S\}$, $P_o = \{x \mid x \in S\}$, $x \models \uparrow y$, if $x \in \uparrow y$, then $P = (P_o, \models, P_a)$ is a Chu space.

(1) $\{\uparrow i \mid i \leq 0, \text{ or } i \geq 0\} \cup \{\uparrow b\}$ is an approximable concept, not a concept.

(2) Clearly $* \in \alpha(\omega(\{\uparrow i \mid i \geq 0\}))$, and but for any finite subset $X$ of $\{\uparrow i \mid i \geq 0\}$, we have $* \notin \alpha(\omega(X))$. $\{\uparrow i \mid i \geq 0\}$ is a family of concept.

## 4 Approximable Concepts in Fuzzy Setting

In [1, 5], suppose $X$ and $Y$ are two sets with **L**-equalities $\approx_X$ and $\approx_Y$, respectively; $I$ an **L**-relation between $X$ and $Y$ which is compatible with respect to $\approx_X$ and $\approx_Y$. A pair $\langle ^\uparrow, ^\downarrow \rangle$ of mappings was defined as:

$^\uparrow : L^X \to L^Y$, for $A \in L^X$, $A^\uparrow(y) = \bigwedge\limits_{x \in X} A(x) \to I(x, y)$.

and $^\downarrow : L^Y \to L^X$, for $B \in L^Y$, $B^\downarrow(x) = \bigwedge\limits_{y \in Y} B(y) \to I(x, y)$.

Then $\langle X, Y, I \rangle$ is a formal **L**-context; $\langle A, B \rangle$ is a concept in fuzzy setting, if $A = A^{\uparrow\downarrow}$, $B = B^{\downarrow\uparrow}$. That is, $A$ is an extent of a concept, $B$ is an intent of a concept; or $A$ is a concept of objects, $B$ is a concept of attributes. $\beta(X, Y, I) = \{\langle A, B \rangle \mid \langle A, B \rangle \text{ is a concept }\}$ is a formal concept lattice.

As a generalization, we introduced the notion of an approximable concept in fuzzy setting ([9]). According to the reviewer's suggestion, there exist two choices for the definition of an approximable concept. We adopted one kind (Definition 4) in [9]. In the section, we discuses the other kind (Definition 5), and prove the equivalence between the two definitions.

Given two **L**-ordered sets $(X, \approx_X)$, $(Y, \approx_Y)$, and $I$ is an **L**-relation. Let $P_o = (X, \approx_X)$, $P_a = (Y, \approx_Y)$, and $\models$ induced by the **L**-relation $I$, that is to say, $(x \models y) = (xIy)$. We obtain a Chu space $P = ((X, \approx_X), \models, (Y, \approx_Y))$ in fuzzy setting, and $\alpha =^{\uparrow}$, $\omega =^{\downarrow}$, i.e.,

$$\alpha : L^X \to L^Y, \text{ for } A \in L^X, \ \alpha(A)(y) = A^{\uparrow}(y) = \bigwedge_{x \in X} A(x) \to I(x, y).$$

$$\omega : L^Y \to L^X, \text{ for } B \in L^Y, \ \omega(B)(x) = B^{\downarrow}(x) = \bigwedge_{y \in Y} B(y) \to I(x, y).$$

**Definition 3.** Suppose $H \in L^X$, if $\{x \in X \mid H(x) > 0\}$ is finite, then $H$ is called finite.

Clearly if **L**=**2**, $\{x \in X \mid H(x) > 0\} = \{x \in X \mid H(x) = 1\}$, is the same with the finite set in classical set theory.

In [9], we defined the notion of an approximable concept,

**Definition 4.** Given $A \in L^X$, if for each finite $H \in L^X$, we have $(H \preceq A) \leq (\omega(\alpha(H)) \preceq A)$, i.e., $S(H, A) \leq S(\omega(\alpha(H)), A)$, then $A$ is called to be an extent of a formal fuzzy approximable concept. $A$ is also called an (formal fuzzy) approximable concept (of objects).

Dually, a set $A \in L^Y$ is an intent of a formal fuzzy approximable concept, if for each finite $H \in L^Y$, we have $(H \preceq A) \leq (\alpha(\omega(H)) \preceq A)$, i.e., $S(H, A) \leq S(\alpha(\omega(H)), A)$. $A$ is also called an (formal fuzzy) approximable concept (of attributes). We will use the symbol $\mathcal{A}\ (Y, I)$ to denote the set of all approximable concepts $A$ (of attributes).

Since for each finite $H \in L^X$, we have $H \subseteq \omega(\alpha(H))$, that is to say, $H(x) \leq \omega(\alpha(H))(x)$ for every $x \in X$. So we obtain $H(x) \to A(x) \geq \omega(\alpha(H))(x) \to A(x)$. Thus $S(H, A) \geq S(\omega(\alpha(H)), A)$ for every $A \in L^X$.

In the similar way, for each finite $H \in L^Y$, and $A \in L^Y$, we also have $S(H, A) \geq S(\alpha(\omega(H)), A)$.

By the above discussion, we obtain an equivalent definition,

**Definition 4$'$.** $A \in L^X$ is called an extent of a formal fuzzy approximable concept, if for each finite $H \in L^X$, we have $(H \preceq A) = (\omega(\alpha(H)) \preceq A)$, i.e., $S(H, A) = S(\omega(\alpha(H)), A)$.

Dually, an **L**-set $A \in L^Y$ is called an intent of a formal fuzzy approximable concept, if for each finite $H \in L^Y$, we have $(H \preceq A) = (\alpha(\omega(H)) \preceq A)$, i.e., $S(H, A) = S(\alpha(\omega(H)), A)$.

The second choice for the definition of an approximable concept is Definition 5.

**Definition 5.** Given $A \in L^X$, if for each finite $H \in L^X$, and $H \subseteq A$, we have $\omega(\alpha(H)) \subseteq A$, then $A$ is called to be an extent of a formal fuzzy approximable concept. $A$ is also called an (formal fuzzy) approximable concept (of objects).

Dually, a set $A \in L^Y$ is an intent of a formal fuzzy approximable concept, if for each finite $H \in L^Y$, and $H \subseteq A$, we have $\alpha(\omega(H)) \subseteq A$. $A$ is also called an (formal fuzzy) approximable concept (of attributes).

From the one direction, we have

**Lemma 2.** Suppose $A$ is an approximable concept in the sense of Definition 4, then $A$ is also an approximable concept in the sense of Definition 5.

**Proof.** It is clearly.

From the other direction, suppose $A$ is an approximable concept in the sense of Definition 5, for $F = \{A(y_0)/y_0\}$, clearly we have $F \subseteq A$. Thus $\alpha(\omega(F)) \subseteq A$. So we obtain $\alpha(\omega(F))(y) \leq A(y)$ for every $y \in Y$.

$\omega(F)(x) = \bigwedge_{y \in Y} F(y) \to I(x, y)$

$= A(y_0) \to I(x, y_0)$.

and

$\alpha(\omega(F))(y) = \bigwedge_{x \in X} \omega(F)(x) \to I(x, y)$

$= \bigwedge_{x \in X} (A(y_0) \to I(x, y_0)) \to I(x, y)$.

By Definition 5, we have,

$\bigwedge_{x \in X} (A(y_0) \to I(x, y_0)) \to I(x, y) \leq A(y)$.

**Lemma 3.** Suppose $A$ is an approximable concept in the sense of Definition 5, then $A$ is also an approximable concept in the sense of Definition 4.

**Proof.** Suppose $A$ is an approximable concept in the sense of Definition 5.

(1) For simplicity, we may assume $H = \{a/y_0\}$, then

$S(H, A) = \bigwedge_{y \in Y} H(y) \to A(y) = a \to A(y_0)$.

$\omega(H)(x) = \bigwedge_{y \in Y} H(y) \to I(x, y)$

$= a \to I(x, y_0)$.

and

$\alpha(\omega(H))(y) = \bigwedge_{x \in X} \omega(H)(x) \to I(x, y)$

$= \bigwedge_{x \in X} [a \to I(x, y_0)] \to I(x, y)$.

(2) We have to prove $A$ is an approximable concept in the sense of Definition 4, it suffices to prove $S(H, A) \leq S(\alpha(\omega(H)), A)$. i.e.,

$a \to A(y_0) \leq \bigwedge_{y \in Y} [\bigwedge_{x \in X} (a \to I(x, y_0)) \to I(x, y)] \to A(y) \quad (*)$.

(3) To prove (*), it suffices to prove

$a \to A(y_0) \leq [\bigwedge_{x \in X} (a \to I(x, y_0)) \to I(x, y)] \to A(y)$ holds for every $y \in Y$,

It is valid, since

$a \to A(y_0)$

$\leq [A(y_0) \to I(x, y_0)] \to [a \to I(x, y_0)]$

$\leq [(a \to I(x, y_0)) \to I(x, y)] \to [(A(y_0) \to I(x, y_0)) \to I(x, y)]$,

thus, we obtain,

$[(a \to I(x, y_0)) \to I(x, y)] \otimes [a \to A(y_0)] \leq [(A(y_0) \to I(x, y_0)) \to I(x, y)]$,

so we have

$$[(a \to I(x,y_0)) \to I(x,y)] \otimes [a \to A(y_0)] \le \bigwedge_{x\in X} [(A(y_0) \to I(x,y_0)) \to I(x,y)],$$

that is to say,

$$\begin{aligned}&[a \to A(y_0)]\\ &\le [(a \to I(x,y_0)) \to I(x,y)] \to \bigwedge_{x\in X} [(A(y_0) \to I(x,y_0)) \to I(x,y)]\\ &\le \bigwedge_{x\in X} [(a \to I(x,y_0)) \to I(x,y)] \to \bigwedge_{x\in X} [(A((y_0) \to I(x,y_0)) \to I(x,y)]\\ &\le \bigwedge_{x\in X} [(a \to I(x,y_0)) \to I(x,y)] \to A(y).\end{aligned}$$

By this, (*) holds. Hence we obtain $S(H,A) \le S(\alpha(\omega(H)),A)$. That is, $A$ is an approximable concept in the sense of Definition 4.

So we obtain,

**Proposition 1.** Definition 4 and Definition 5 are equivalent.

The following proposition gives a representation of an approximable concept.

**Proposition 2.** Suppose $A$ is an approximable concept in $\mathcal{A}(Y.I)$, then

$$A(y) = \bigvee_{\text{finite } H\in L^Y} S(H,A) \otimes \alpha(\omega(H))(y).$$

**Proof.** Since $A$ is an approximable concept, so for each finite $H \in L^Y$, we have $S(H,A) \le S(\alpha(\omega(H)),A)$. Thus we have,

$$S(H,A) \otimes \alpha(\omega(H))(y) \le S(\alpha(\omega(H)),A) \otimes \alpha(\omega(H))(y) \le A(y).$$

So we obtain,

$$\bigvee_{\text{finite } H\in L^Y} S(H,A) \otimes \alpha(\omega(H))(y) \le A(y).$$

On the other hand, for $\{A(y)/y\} \in L^Y$, since $H \subseteq \alpha(\omega(H))$, we obtain

$$S(\{A(y)/y\},A) \otimes \alpha(\omega(\{A(y)/y\}))(y) = 1 \otimes \alpha(\omega(\{A(y)/y\}))(y)$$
$$= \alpha(\omega(\{A(y)/y\}))(y) \ge \{A(y)/y\}(y) = A(y).$$

Furthermore we have,

$$\bigvee_{\text{finite } H\in L^Y} S(H,A) \otimes \alpha(\omega(H))(y) \ge A(y).$$

Hence the equation holds.

**Note**. In [10], we introduced the notions of a directed set, a way-below relation, a continuous lattice, an algebraic lattice in fuzzy setting. In [9], we adopted Definition 4, and showed that approximable concept lattices represent algebraic completely lattice **L**-ordered sets in the sense of [3]. For more details, see [8, 9, 10, 11].

## 5 Generalized Approximable Concepts

As showed in Introduction, R. Bělohlávek and Stanislav Krajči gave the generalization of concept lattice, respectively, see [1, 6].

In [16, 17], Stanislav Krajči obtained a common platform for both of them, and proved all complete lattices are isomorphic to the generalized concept lattices.

We introduce some main notions [16, 17].

Suppose $L$ is a poset, $C$ and $D$ are two supremum-complete upper-semilattices. i.e., there exists $\sup X = \bigvee X$ for each subset of $C$ or $D$ (in fact, $C, D$ are complete lattices). Let $\bullet : C \times D \to L$ be monotone and left-continuous in both their arguments, that is to say,

1a) $c_1 \le c_2$ implies that $c_1 \bullet d \le c_2 \bullet d$ for all $c_1, c_2 \in C$ and $d \in D$.

1b) $d_1 \le d_2$ implies that $c \bullet d_1 \le c \bullet d_2$ for all $c \in C$ and $d_1, d_2 \in D$.

2a) If $c \bullet d \le \iota$ holds for $d \in D, \iota \in L$ and for all $c \in X \subseteq C$, then $\sup X \bullet d \le l$.

2b) If $c \bullet d \le \iota$ holds for $c \in C, \iota \in L$ and for all $d \in Y \subseteq D$, then $c \bullet \sup Y \le l$.

Let $A$ and $B$ be non-empty sets and $R$ be $L$-fuzzy relation on their Cartesian product, $R : A \times B \to L$. Stanislav Krajči defined two mappings as follows,

(1) $\nearrow$: ${}^B D \to^A C$, if $g : B \to D$, then $\nearrow (g) : A \to C$, where $\nearrow (g)(a) = \sup\{c \in C \mid \forall b \in B, c \bullet g(b) \le R(a, b)\}$.

(2) $\swarrow$: ${}^A C \to^B D$, if $f : A \to C$, then $\swarrow (f) : B \to D$, where $\swarrow (f)(b) = \sup\{d \in D \mid \forall a \in A, f(a) \bullet d \le R(a, b)\}$.

In [16, 17], Stanislav Krajči introduced a generalized concept lattice.

Based on the common platform, we give a generalization of an approximable concept, i.e., a generalized approximable concept.

The notions of a directed set, an algebraic lattice were introduced in Section 2.1. In the section, because the definition is not symmetric, similarly, we also give the notions of a up-directed set, a left-algebraic lattice.

**Definition 6** Suppose $h : B \to D$, if there exists $\{b_i \mid i \in I\} \subseteq B$, where $I$ is a finite index, such that $h(b_i) \neq 0$, and $h(b) = 0$ for all $b \in B, b \neq b_i$, then $h$ is called finite.

**Definition 7** Suppose $g : B \to D$, $g$ is a generalized approximable concept, if for each finite $h \le g$, we have $\swarrow\nearrow (h) \le g$.

The collection of all generalized approximable concepts denoted by $\mathcal{A}$. In the first part, we will show that $(\mathcal{A}, \le)$ is a left-algebraic lattice.

When $L, C, D$ are finite, the notions of a generalized approximable concept and a generalized concept are identical.

**Lemma 4** Suppose $g \in \mathcal{A}$, $\{\swarrow\nearrow (h) \mid \text{finite } h \le g\}$ is up-directed.

**Proof** For $g \in \mathcal{A}$, suppose $h_1, h_2$ are finite, and $h_1, h_2 \le g$, we have $\swarrow\nearrow (h_1) \le \swarrow\nearrow (h_1 \vee h_2)$, $\swarrow\nearrow (h_2) \le \swarrow\nearrow (h_1 \vee h_2)$. where $(h_1 \vee h_2)(a) = h_1(a) \vee h_2(a)$. Thus $h_1 \vee h_2$ is also finite, and $h_1 \vee h_2 \le g$.

**Lemma 5** Suppose $g \in \mathcal{A}$, we have $g = \sup\{\swarrow\nearrow (h) \mid \text{finite } h \le g\}$.

**Proof** It is trivial.

By Lemmas 4, 5, we have $g$ is the supremum of a up-directed set.

**Lemma 6** Suppose $h$ is finite, then $\swarrow\nearrow (h)$ is compact.

**Proof** It is trivial.

**Lemma 7** Suppose $\{g_i \mid i \in I\}$ is a up-directed set of generalized approximable concepts, then $\bigvee\limits_{i \in I} g_i$ is also a generalized approximable concept.

**Proof** For each finite $h \le \bigvee\limits_{i \in I} g_i$, there exist $g_1, g_2, \cdots, g_m$, such that $h(b_i) \le g_i(b_i)$; and for every $b \in B$, $b \neq b_i$, $h(b) = 0$.

Since $\{g_i \mid i \in I\}$ is up-directed, there exists $g_{i_0}$, such that $h \le g_{i_0}$. Furthermore, $g_{i_0}$ is a generalized approximable concept, we have $\swarrow\nearrow(h) \le g_{i_0} \le \bigvee_{i\in I} g_i$, which implies that $\bigvee_{i\in I} g_i$ is a generalized approximable concept.

**Lemma 8** Suppose $\{g_i \mid i \in I\}$ is a set of generalized approximable concepts, then $\bigwedge_{i\in I} g_i$ is also a generalized approximable concept.

**Proof** It is trivial.

**Proposition 3.** $(\mathcal{A}, \le)$ is left-algebraic.

**Proof** By Lemmas 4, 5, 6, 7, 8.

Proposition 3 shows that all generalized approximable concepts form a left-algebraic lattice. Conversely, in the second part, suppose $(P, \le)$ is a left-algebraic lattice, we will construct a generalized approximable concept lattice which is isomorphic to $(P, \le)$.

The elements of $P$ denoted by $x, y$, and the elements of $K(\ll)$ denoted by $p, q$, where $K(\ll)$ is the set of all compact elements.

Let $A = P$, $B = K(\ll)$, and $R(x, p) : A \times B \to L$ indicates the degree of $p$ belonging to $x$. By Proposition 3, we obtain a generalized approximable concept lattice $(\mathcal{A}, \le)$.

In what follows, We will prove that $(P, \le)$ is isomorphic to $(\mathcal{A}, \le)$.

Suppose $e \in D, p \in K(\ll)$, we define a mapping $\{e/p\} : K(\ll) \to D$, where $(\{e/p\})(p) = e$; $(\{e/p\})(q) = 0$, if $q \ne p$.

Similarly, for $m \in C$, $x \in P$, we also define a mapping $\{m/x\} : P \to C$, where $\{m/x\}(x) = m$; and $\{m/x\}(y) = 0$, if $y \ne x$.

**Lemma 9** (1) $\nearrow(\{e/p\})(x) = \sup\{c \in C \mid c \bullet e \le R(x, p)\}$,

(2) $\swarrow(\{m/x\})(p) = \sup\{d \in D \mid m \bullet d \le R(x, p)\}$.

**Proof** (1) $\nearrow(\{e/p\})(x)$

$= \sup\{c \in C \mid \forall q \in K(\ll), c \bullet (\{e/p\})(q) \le R(x, q)\}$

$= \sup\{c \in C \mid c \bullet e \le R(x, p)\}$.

(2) It is analogous.

**Proposition 4.** Suppose $g : K(\ll) \to D$ is a generalized approximable concept, $p \in K(\ll)$, $g(p) \ne 0$, then we have $g(p) = 1$.

**Proof** For $g : K(\ll) \to D$, and $p \in K(\ll)$, $g(p) \ne 0$. Let $e = g(p) \in D$, we obtain a mapping $\{e/p\} : K(\ll) \to D$ as defined above.

By Lemma 9, let $x = p \in K(\ll) \subseteq P$, we have

$\nearrow(\{e/p\})(p)$

$= \sup\{c \in C \mid c \bullet e \le R(p, p)\} = 1$.

$\swarrow\nearrow(\{e/p\})(p)$

$= \sup\{d \mid \nearrow(\{e/p\})(p) \bullet d \le R(p, p)\} = 1$.

Since $\{d/p\} \le g$, and $g$ is a generalized approximable concept, we have $\swarrow\nearrow(\{e/p\}) \le g$. Thus $\swarrow\nearrow(\{e/p\})(p) \le g(p)$. So we obtain $g(p) = 1$.

By this, for a generalized approximable concept $g$, we define

$x_g = \vee\{p \mid g(p) = 1\}$.

On the other hand, for every $x \in P$, since $P$ is left-algebraic,

$x = \vee\{\downarrow x \cap K(\ll)\}$,

we may define $g_x : K(\ll) \to D$, such that $g_x(p) = 1$ for every $p \ll x$. Then $g_x$ is a generalized approximable concept. Thus we obtain an isomorphism between $P$ and generalized approximable concept lattice $\mathcal{A}$. Thus $P$ and $\mathcal{A}$ is isomorphic.

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