Unique Factorization Theorem and Formal Concept Analysis

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Abstract. In the theory of generalised colourings of graphs, the Unique Factorization Theorem (UFT) for additive induced-hereditary properties of graphs provides an analogy of the well-known Fundamental Theorem of Arithmetics. The purpose of this paper is to present a new, less complicated, proof of this theorem that is based on Formal Concept Analysis. The method of the proof can be successfully applied even for more general mathematical structures known as relational structures.

1 Introduction and motivation

Formal Concept Analysis (briefly FCA) is a theory of data analysis which identifies conceptual structures among data sets. It was introduced by R. Wille in 1982 and since then has grown rapidly (for a comprehensive overview see [12]). The mathematical lattices that are used in FCA can be interpreted as classification systems. Formalized classification systems can be analysed according to the consistency of their relations. Some extensions and modifications of FCA can be found e.g. in [16].

In this paper we provide a new proof of the Unique Factorization Theorem (UFT) for induced-hereditary additive properties of graphs. The problem of unique factorization of reducible hereditary properties of graphs into irreducible factors was formulated as Problem 17.9 in the book [15] of T.R. Jensen and B. Toft. Our proof is significantly shorter as the previous ones and it is based on FCA. Moreover, FCA allows us to work with concepts instead of graphs and the

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reader can rather easily see that using this approach we can prove UFT even for properties of more general structures like hypergraphs, coloured hypergraphs, posets, etc. Such general mathematical object are very often called relational structures.

In general, we follow standard graph terminology (see e.g. [1]). In particular, we denote by $\mathbb{N}$ the set of positive integers and by $\mathcal{I}^\omega$, $\mathcal{I}$ and $\mathcal{I}^{conn}$ the class of all simple countable graphs, simple finite graphs and simple finite connected graphs, respectively and K_n stands for the complete graph of order n. For a positive integer k and a graph G, the notation $k.G$ is used for the union of k vertex disjoint copies of G. The *join* of graphs G, H is the graph obtained from the disjoint union G and H by joining all vertices of G with all the vertices of H.

All our considerations can be done for arbitrary infinite graphs, however, in order to avoid formal set-theoretical problems, we shall consider only countable infinite graphs. Moreover, we assume that the vertex set $V(G)$ of a graph G is a subset of a given countable set, say U. A graph property $\mathcal{P}$ is any isomorphism-closed nonempty subclass of $\mathcal{I}^\omega$. It means that investigating graph properties, in principle, we restrict our considerations to unlabeled graphs.

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be graph properties. A vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-*colouring* (*partition*) of a graph $G = (V, E)$ is a partition $(V_1, V_2, \ldots, V_n)$ of $V(G)$ (every pair of V_i's has empty intersection and the union of V_i's forms V) such that each colour class V_i induces a subgraph $G[V_i]$ having property $\mathcal{P}_i$. For convenience, we allow empty partition classes in the partition sequence. An empty class induces the null graph $K_0 = (\emptyset, \emptyset)$. If each of the $\mathcal{P}_i$'s, $i = 1, 2, \ldots, n$, is the property $\mathcal{O}$ of being edgeless, we have the well-known proper vertex n-colouring. A graph G which have a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colouring is called $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-*colourable*, and in such a situation we say that G has property $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$. For more details concerning generalized graph colourings we refer the reader to [2, 3, 15].

In 1951, de Bruijn and Erdős proved that an infinite graph G is k-colourable if and only if every finite subgraph of G is k-colourable. An analogous compactness theorem for generalized colourings was proved in [7]. The key concept for the Vertex Colouring Compactness Theorem of [7] is that of a property being of *finite character*. Let $\mathcal{P}$ be a graph property, $\mathcal{P}$ is of *finite character* if a graph in $\mathcal{I}^\omega$ has property $\mathcal{P}$ if and only if each its finite induced subgraph has property $\mathcal{P}$. It is easy to see that if $\mathcal{P}$ is of finite character and a graph has property $\mathcal{P}$ then so does every induced subgraph. A property $\mathcal{P}$ is said to be *induced-hereditary* if $G \in \mathcal{P}$ and $H \leq G$ implies $H \in \mathcal{P}$, that is $\mathcal{P}$ is closed under taking induced subgraphs. Thus properties of finite character are induced-hereditary. However not all induced-hereditary properties are of finite character; for example the graph property $\mathcal{Q}$ of not containing a vertex of infinite degree is induced-hereditary but not of finite character. Let us also remark that every property which is hereditary with respect to every subgraph (we say simply *hereditary*) is induced-hereditary as well. The properties of being edgeless, of maximum degree at most k, K_n-free, acyclic, complete, perfect, etc. are properties of finite character. The compactness theorem for $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colourings, where the $\mathcal{P}_i$'s are of finite character, have been proved using Rado's Selection Lemma.

Theorem 1 (Vertex Colouring Compactness Theorem, [7]**).** *Let G be a graph in $\mathcal{I}^\omega$ and let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be properties of graphs of finite character. Then G is $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colourable if every finite induced subgraph of G is $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colourable.*

Let us denote by $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n,\ n \geq 2$ the set of all $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colourable graphs. The binary operation $\circ$ is obviously commutative, associative on the class of graph properties and $\Theta = \{K_0\}$ is its neutral element. The properties $\Theta, \mathcal{I}$ and $\mathcal{I}^\omega$ are said to be trivial. A nontrivial graph property $\mathcal{P}$ is said to be *reducible* if there exist nontrivial graph properties $\mathcal{P}_1, \mathcal{P}_2$, such that $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$; otherwise $\mathcal{P}$ is called *irreducible*. In what follows each property is considered to be nontrivial.

The problem of unique factorization of a reducible induced-hereditary property into induced-hereditary factors was introduced in connection with the study of the existence of uniquely colourable graphs with respect to hereditary properties (see [2, 3] and Problem 17.9. in the book [15]). In general, there are only few graph properties that have a unique factorization into irreducible ones (see [8, 10]). However, for some important classes of graph properties the Unique Factorization Theorems can be proved. In [19] it is proved that every reducible property of finite graphs, which is closed under taking subgraphs and disjoint union of graphs (such properties are called *additive*) is uniquely factorisable into irreducible additive hereditary factors. An analogous result was obtained in [10, 17] for additive induced-hereditary properties of finite graphs. Following [2] let us denote by $\mathbb{M}^a$ the set of all additive induced-hereditary properties of finite graphs. Then UFT can be stated as follows.

Theorem 2 (Unique Factorization Theorem - UFT, [10, 17]**).** *Every additive induced-hereditary property of finite graphs is in $\mathbb{M}^a$ uniquely factorisable into a finite number of irreducible additive induced-hereditary properties, up to the order of factors.*

Let us remark, that using Theorem 1 we can prove UFT for the class $\mathbb{M}^{\omega a}$ of the additive properties of infinite (countable) graphs of finite character (see [13]). The proof of the Unique Factorization Theorem is rather complicated. The problems concerning the proof were discussed from different points of view in several papers [6, 10, 11, 13, 17] and in details in PhD thesis (see e.g. [8]). On the other hand, the Theorem 2 has several deep applications related to the existence of uniquely partitionable graphs (see [4, 5]) and consequently the complexity of generalized colourings. A. Farrugia in [9] proved that if $\mathcal{P}$ and $\mathcal{Q}$ are additive induced-hereditary graph properties, then $(\mathcal{P}, \mathcal{Q})$-colouring is NP-hard, with the sole exception of graph 2-colouring (the case where both $\mathcal{P}$ and $\mathcal{Q}$ are the set $\mathcal{O}$ of finite edgeless graphs). Moreover, $(\mathcal{P}, \mathcal{Q})$-colouring is NP-complete if and only if $\mathcal{P}$- and $\mathcal{Q}$-recognition are both in NP. It shows that additive induced-hereditary properties are rather complicated mathematical structures.

The aim of this paper is to present a new method of the proof of the Unique Factorization Theorem, which will eliminate some technical difficulties in the previous proofs. Moreover it shows a new utilisation of the methods of FCA.

2 Hereditary graph properties in the language of FCA

It is quite easy to prove that the sets $\mathbb{M}^a$ ($\mathbb{M}^{\omega a}$) of all additive and induced-hereditary graph properties of finite graphs (of finite character), partially ordered by set inclusion, forms a complete distributive lattice. The lattices of hereditary graph properties have been studied intensively, references may be found in [2, 14, 18]. In this section we will present a new approach to the lattice of additive induced-hereditary graph properties.

In order to proceed we need to introduce some concepts of FCA according to a fundamental book of B. Ganter and R. Wille [12].

Definition 1. *A* **formal context** $\mathbb{K} := (O, M, I)$ *consists of two sets O and M and a relation I between O and M. The elements of O are called the* **objects** *and the elements of M are called the* **attributes** *of the context.*

For a set $A \subseteq O$ of objects we define

$$A' := \{m \in M : gIm \text{ for all } g \in A\}.$$

Analogously, for a set B of attributes we define

$$B' := \{g \in O : gIm \text{ for all } \in B\}.$$

A **formal concept** *of the context (O, M, I) is a pair (A, B) with $A \subseteq O, B \subseteq M, A' = B$ and $B' = A$.*

We call A the **extent** *and B the* **intent** *of a concept (A, B). $\mathbb{L}(O, M, I)$ denotes the set of all concepts of the context (O, M, I).*

If (A_1, B_1) and (A_2, B_2) are concepts of a context and $A_1 \subseteq A_2$ (which is equivalent to $B_2 \subseteq B_1$), we write $(A_1, B_1) \leq (A_2, B_2)$.

For an object $g \in O$ we write $g' = \{m \in M | gIm\}$ and γg for the **object concept** *(g'', g'), where $g'' = \{\{g\}'\}'$.*

Let us mention that, by the Basic Theorem on Concept Lattices, the set $\mathbb{L}(O, M, I)$ of all concepts of the context (O, M, I) partially ordered by the relation $\leq$ (see Definition 1) is a complete lattice.

Let us present additive induced-hereditary graph properties as concepts in a given formal context. Using FCA we can proceed in the following way. Let us define a context (O, M, I) by setting objects to countable simple graphs, e.g. $O = \mathcal{I}^\omega$. For each connected finite simple graph $F \in \mathcal{I}$ let us consider an attribute m_F : "do not contain an induced-subgraph isomorphic to F". Thus GIm_F means that the graph G does not contain any induced subgraph isomorphic to F. We can immediately observe the following:

Lemma 1. *Let $O = \mathcal{I}^\omega$ and $M = \{m_F, F \in \mathcal{I}^{conn}\}$. Then the concepts of the context $\mathbb{K} = (O, M, I)$ are additive induced-hereditary graph properties of finite character and the concept lattice $(\mathbb{L}(O, M, I), \leq)$ is isomorphic to the lattice $(\mathbb{M}^{\omega a}, \subseteq)$. Moreover, for each concept $\mathcal{P} = (A, B)$ there is an object - a countable graph $G \in O$ such that $\mathcal{P} = \gamma G = (G'', G')$.*

Proof. It is easy to verify that the extent of any concept (A, B) of $\mathbb{K}$ forms an additive induced-hereditary property $\mathcal{P} = A$ of finite character. Obviously, each countable graph $G = (V, E)$ in the context $\mathbb{K}$ leads to an "object concept" $\gamma G = (G'', G')$. On the other hand, because of additivity, the disjoint union of all finite graphs having a given additive induced-hereditary property $\mathcal{P} \in \mathbb{M}^{\omega a}$ is a countable infinite graph K satisfying $\gamma K = (\mathcal{P}, \mathcal{I}^{conn} - \mathcal{P})$. □

In order to characterise additive induced-hereditary properties of finite graphs, mainly two different approaches were used: a characterization by generating sets and/or by minimal forbidden subgraphs (see [2] and [11]). While the extent A of a concept $(A, B) \in \mathbb{L}(O, M, I)$ is related to a graph property $\mathcal{P}$, the intent B consists of forbidden connected subgraphs of $\mathcal{P}$. The set $\boldsymbol{F}(\mathcal{P})$ of *minimal forbidden subgraphs* for $\mathcal{P}$ consists of minimal elements of the poset $(B, \leq)$. For a given countable graph $G \in \mathcal{I}^\omega$ let us denote by $age(G)$ the class of all finite graphs isomorphic to finite induced-subgraph of G (see e.g. [20]). Scheinerman in [21] showed, that for each additive induced-hereditary property $\mathcal{P}$ of finite graphs, there is an infinite countable graph G such that $\mathcal{P} = age(G)$. This result corresponds to the proof of Lemma 1. On the other hand, it is worth to mention that $\gamma G = (\mathcal{P}, G')$ does not imply, in general, that $\mathcal{P} = age(G)$. Let us define a binary relation $\cong$ on $\mathcal{I}^\omega$ by $G_1 \cong G_2$ whenever $\gamma G_1 = \gamma G_2$ in the context $\mathbb{K}$, and we say that G_1 is congruent with G_2 with respect to $\mathbb{K}$. Obviously, $\cong$ is an equivalence relation on $\mathcal{I}^\omega$. The aim of the next section is to find appropriate representatives of congruence classes and to describe their properties.

3 Uniquely decomposable graphs

All the previous proofs of UFT are based on a construction of *uniquely $\mathcal{R}$-decomposable* graphs that are defined as follows.

Definition 2. *For given (finite or infinite) graphs $G_1, G_2, \ldots, G_n$, $n \geq 2$, denote by $G_1 * G_2 * l \cdots * G_n$ the set of graphs*

$$\left\{ H \in \mathcal{I}^\omega : \bigcup_{i=1}^{n} G_i \subseteq H \subseteq \sum_{i=1}^{n} G_i \right\},$$

where $\bigcup_{i=1}^{n} G_i$ denotes the disjoint union and $\sum_{i=1}^{n} G_i$ the join of the graphs $G_1, G_2, \ldots, G_n$, respectively. For a graph G, $s \geq 2$, $s * G$ stands for the class $G * G * \cdots * G$, with s copies of G.

Let G be a graph and $\mathcal{R}$ be an additive induced-hereditary property of graphs. Then we put $dec_{\mathcal{R}}(G) = \max\{n :$ there exist a partition $\{V_1, V_2, \ldots, V_n\}$, $V_i \neq$

$\emptyset$, of $V(G)$ such that for each $k \geq 1$, $k.G[V_1] * k.G[V_2] * \cdots * k.G[V_n] \subseteq \mathcal{R}\}$. If $G \notin \mathcal{R}$ we set $dec_{\mathcal{R}}(G)$ to zero.

A graph G is said to be **$\mathcal{R}$-decomposable** *if $dec_{\mathcal{R}}(G) \geq 2$; otherwise G is $\mathcal{R}$-indecomposable.*

A graph $G \in \mathcal{P}$ is called **$\mathcal{P}$-strict** *if $G * K_1 \not\subseteq \mathcal{P}$. The class of all $\mathcal{P}$-strict graphs is denoted by $S(\mathcal{P})$. Put $dec(\mathcal{R}) = \min\{dec_{\mathcal{R}}(G) : G \in S(\mathcal{R})\}$.*

A $\mathcal{R}$-strict graph G with $dec_{\mathcal{R}}(G) = dec(\mathcal{R}) = n \geq 2$ is said to be **uniquely $\mathcal{R}$-decomposable** *if there exists exactly one $\mathcal{R}$-partition $\{V_1, V_2, \ldots, V_n\}$, $V_i \neq \emptyset$, such that for each $k \geq 1$, $k.G[V_1] * k.G[V_2] * \cdots * k.G[V_n] \subseteq \mathcal{R}$. We call the graphs $G[V_1], G[V_2], \ldots, G[V_n]$* **ind-parts** *of the uniquely decomposable graph G.*

These notions are motivated by the following observation: Let us suppose that $G \in \mathcal{R} = \mathcal{P} \circ \mathcal{Q}$ and let (V_1, V_2) be a $(\mathcal{P}, \mathcal{Q})$-partition of G. Then by additivity of $\mathcal{P}$ and $\mathcal{Q}$ we have that $k.G[V_1] * k.G[V_2] \subseteq \mathcal{R}$ for every positive integer k. Thus, if the property $\mathcal{R}$ is reducible, every graph $G \in \mathcal{R}$ with at least two vertices is $\mathcal{R}$-decomposable.

We proved in [13, 17] that for any reducible additive induced-hereditary property also the converse assertion holds:

Theorem 3. *An induced-hereditary additive property $\mathcal{R}$ is reducible if and only if all graphs in $\mathcal{R}$ with at least two vertices are $\mathcal{R}$-decomposable.*

Remark that almost all graphs in $\mathcal{R}$ are $\mathcal{R}$-strict and each graph $G \in \mathcal{R}$ is an induced subgraph of a $\mathcal{R}$-strict graph. To present our main result we need some notions from [10]:

Definition 3. *Let $d_0 = \{U_1, U_2, \ldots, U_m\}$ be a $\mathcal{P}$-partition of a graph $G \in \mathcal{P}$. A $\mathcal{P}$-partition $d_1 = \{V_1, V_2, \ldots, V_n\}$ of G* **respects** *d_0 if no V_i intersects two or more U_j's; that is each V_i is contained in some U_j. We say that the graph $G^* \in s * G$* **respects** *d_0 if $G^* \in s.G[U_1] * s.G[U_2] * \cdots * s.G[U_m]$. For a graph $G^* \in s * G$, denote the copies of G by $G^1, G^2, \ldots, G^s$. Then we say that a $\mathcal{P}$-partition $d = \{V_1, V_2, \ldots, V_n\}$ of G^** **respects** *d_0* **uniformly** *whenever for each V_i there is a U_j such that for every G^k, $V_i \cap V(G^k) \subseteq U_j$.*

If G is uniquely $\mathcal{R}$-decomposable, its ind-parts respect d_0 if its unique $\mathcal{R}$-partition respects d_0. If G^* is uniquely $\mathcal{R}$-decomposable, it ind-parts respect d_0 uniformly if for some s the graph $G^* \in s * G$ respects d_0 and the unique $\mathcal{R}$-partition of G^* respects d_0 uniformly.

Based on the construction given in [17] A. Farrugia and R.B. Richter proved:

Theorem 4. ([10, 17]) *Let G be an $\mathcal{R}$-strict graph with $dec_{\mathcal{R}}(G) = dec(\mathcal{R}) = n \geq 2$ and let $d_0 = \{U_1, U_2, \ldots, U_m\}$ be a fixed $\mathcal{R}$-partition of G. Then there is a uniquely $\mathcal{R}$-decomposable finite graph $G^* \in s * G$, for some s, that respects d_0, and moreover any $\mathcal{R}$-partition of G^* with n parts respects d_0 uniformly.*

Using Theorem 4 we can prove:

Theorem 5. *Let $\mathcal{R} \in \mathbb{M}^{\omega a}$ be a reducible graph property of finite character. Then there exists a uniquely $\mathcal{R}$-decomposable infinite countable graph H such that $\gamma H = (\mathcal{R}, H')$ and $age(H) = \mathcal{R} \cap \mathcal{I}$.*

Proof. Following E. Scheinerman [21], a *composition sequence* of a class $\mathcal{P}$ of finite graphs is a sequence of finite graphs $H_1, H_2, \ldots, H_n, \ldots$ such that $H_i \in \mathcal{P}, H_i < H_{i+1}$ for all $i \in \mathbb{N}$ and for all $G \in \mathcal{P}$ there exists a j such that $G \leq G^j$. According to Theorem 4, we can easily find a composition sequence $H_1, H_2, \ldots, H_n, \ldots$ of $\mathcal{R} \cap \mathcal{I}$ consisting of finite uniquely $\mathcal{R}$-decomposable graphs. Without loss of generality, we may assume that if $i < j$, $i, j \in \mathbb{N}$, then $V(H_i) \subset V(H_j)$. Let $V(H) = \bigcup_{i \in \mathbb{N}} V(H_i)$ and $\{u, v\} \in E(H)$ if and only if $\{u, v\} \in E(H_j)$ for some $j \in \mathbb{N}$. It is easy to see that $age(H) = \mathcal{R} \cap \mathcal{I}$, implying $\gamma H = (\mathcal{R}, H')$. Let us remark that, according to the Theorem 1, H is $\mathcal{R}$-decomposable if every finite induced subgraph of H is $\mathcal{R}$-decomposable. In order to verify, that H is uniquely $\mathcal{R}$-decomposable it is sufficient to verify that if $\{V_{j_1}, V_{j_2}, \ldots, V_{j_n}\}$, $V_{j_i} \neq \emptyset$ is the unique $\mathcal{R}$-partition of H_j, $j \in \mathbb{N}$, then $\{U_1, U_2, \ldots, U_n\}$, where $U_k = \bigcup_{j \in \mathbb{N}} V_{j_k}$, $k = 1, 2, \ldots, n$, is the unique $\mathcal{R}$-partition of H. Indeed, this is because the existence of other $\mathcal{R}$-partition of H would imply the existence of other partition of some H_i and it provides a contradiction. □

4 Unique Factorization Theorem for properties of finite character

In [13], based on Theorem 1 and Theorem 2 we proved:

Theorem 6. *Every reducible additive property $\mathcal{R}$ of finite character is uniquely factorisable into finite number of irreducible factors belonging to $\mathbb{M}^{\omega a}$.*

Here we present a new proof of the Theorem 6 based on the Theorem 5 in the context $\mathbb{K}$.

Proof. According to the Theorem 5, let H be a uniquely $\mathcal{R}$-decomposable infinite countable graph such that $\gamma H = (\mathcal{R}, H')$ and let $d_H = \{W_1, W_2, \ldots, W_n\}$ be the unique $\mathcal{R}$-partition of H. Let $\mathcal{P}_i = \gamma H[W_i]$ for $i = 1, 2, \ldots, n = dec(\mathcal{R})$. Then obviously we have $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$. If there would be some other factorization of $\mathcal{R}$ into n irreducible factors then obviously H would have another $\mathcal{R}$-partition, which contradicts to the fact that H is uniquely $\mathcal{R}$-decomposable. Since $dec(H) = dec(\mathcal{R}) = n$, there is no factorization of $\mathcal{R}$ into more then n factors. Thus to prove that $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$ is the unique factorization of $\mathcal{R}$. Further, let $\mathcal{R} = \mathcal{Q}_1 \circ \mathcal{Q}_2 \circ \ldots \circ \mathcal{Q}_m, m < n$ and $d_0 = \{U_1, U_2, \ldots U_m\}$ be a $(\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_m)$-partition of H. Then, by Theorem 4, there is an $s \in \mathbb{N}$ such that $s * H$ respects d_0 uniformly. Thus, since $m < n$, there exists an index j such that $H[U_j] \in H[W_r] * H[W_s]$, implying $\mathcal{Q}_j$ is reducible. □

5 Conclusion

By a careful examination of the previous considerations and arguments, it is not very difficult to see, that for the presented method of the proof it is not important that we are dealing with simple graphs. Indeed, without any substantial change the presented proofs can be applied for directed graphs, hypergraphs or partially ordered sets. All these mathematical objects are examples of so-called relational structures. Thus we obtain a general UFT that is applicable for additive properties of finite character for different objects, with various applications in computer science. For other details we refer the reader to [6].

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