Closure Systems of Equivalence Relations and Their Labeled Class Geometries

Tim B. Kaiser

Darmstadt University of Technology, 64287 Darmstadt, Germany
tkaiser@mathematik.tu-darmstadt.de

Abstract. The notion of an affine ordered set is specialized to that of a complete affine ordered set, which can be linked to attribute-complete many-valued contexts and is categorically equivalent to the notion of a closed system of equivalence relations (SER). This specialization step enables us to give conditions under which the complete affine ordered set can be interpreted as the set of congruence classes labeled with the congruence relation they stem from yielding a coordinatization theorem for affine ordered sets.

1 Introduction

In [KS04] the notion of affine ordered sets is introduced to provide an order-theoretic, geometric counterpart of (simple) many-valued contexts. Here we specialize the notion of an affine ordered set to that of a complete affine ordered set, which is categorically equivalent to attribute-complete many-valued contexts and to closed systems of equivalence relations (SER). This specialization step enables us to add an algebraic aspect, that is, to give conditions under which the complete affine ordered set can be interpreted as the set of congruence classes of an algebra labeled with the congruence relation they stem from. This approach can be seen in the tradition of coordinatization theorems in geometry where a prominent example is the coordinatization of affine planes via the Theorem of Desargues.

In Section 2 we introduce the notions of attribute-complete many-valued contexts and closed SERs and insinuate the correspondence between the two. The order-theoretic geometric counterpart is introduced as *complete affine ordered set* in Section 3, and Section 4 shows the categorical equivalence between complete affine ordered sets and closed SERs. The second part of the paper, consisting of Section 5, deals with the question how to coordinatize closed SERs and complete affine ordered sets.

2 Attribute-complete Many-valued Contexts and Closed Systems of Equivalence Relations

Data tables can be formalized as many-valued contexts as it is common in Formal Concept Analysis [GW99]. Many-valued contexts are also known as *Chu Spaces* [Pr95] or *Knowledge Representation Systems* [Pa91].

Definition 1 (many-valued context). *A* (complete) many-valued context *is a structure* $\mathbb{K} := (G, M, W, I)$*, where* G *is a set of objects,* M *is a set of attributes,* W *is a set of values and* $I \subseteq G \times M \times W$ *is a ternary relation, where for every* $(g, m) \in G \times M$ *there exists a unique* $w \in W$ *with* $(g, m, w) \in I$*; in the following* m *will be considered as a function from* G *to* W *via* $m(g) := w$*.*

We call an attribute $m \in M$ an *id attribute* if for any two objects $g_1, g_2 \in G$ the values of g_1 and g_2 regarding to m are different (i.e. $m(g_1) \neq m(g_2)$). The following definition provides a notion of dependency between attributes of a many-valued context.

Definition 2 (functional dependence). *If* M_1 *and* M_2 *are sets of attributes of a many-valued context* (G, M, W, I)*, we call* M_2 functionally dependent *on* M_1 *(in symbols:* $M_1 \to M_2$*) if for all pairs of objects* $g, h \in G$

$$\forall m_1 \in M_1 : m_1(g) = m_1(h) \Rightarrow \forall m_2 \in M_2 : m_2(g) = m_2(h).$$

If $M_1 \to M_2$ *and* $M_2 \to M_1$*, the sets of attributes,* M_1 *and* M_2*, are called* functionally equivalent*, denoted by* $M_1 \leftrightarrow M_2$*.*

For a map $f : A \to B$ the *kernel* of f is defined as the equivalence relation $\ker(f) := \{(a, b) \in A^2 \mid f(a) = f(b)\}$. It is easily seen that $M_1 \to M_2$ holds if and only if $\bigcap_{m_1 \in M_1} \ker(m_1) \subseteq \bigcap_{m_2 \in M_2} \ker(m_2)$. Accordingly, m_1 and m_2 are functionally equivalent if and only if $\bigcap_{m_1 \in M_1} \ker(m_1) = \bigcap_{m_2 \in M_2} \ker(m_2)$. Many-valued contexts where any two functionally equivalent attributes are equal will be called *simple*.

Definition 3 (attribute-complete many-valued context). *A many-valued context* $\mathbb{K} := (G, M, W, I)$ *is called* attribute-complete *if it is simple, has an id attribute, and*

$$\forall N \subseteq M \exists m \in M : N \leftrightarrow \{m\}.$$

Following the main scheme from [KS04], we assign a system of equivalence relations to attribute-complete many-valued contexts in order to describe them geometrically and order-theoretically in a later step. We recall the basic definitions for systems of equivalence relations from [KS04]. We denote the *identity relation* on the set X by $\Delta_X := \{(x, x) \mid x \in X\}$.

Definition 4 (system of equivalence relations). *We call* $\mathbb{E} := (D, E)$ *a* system of equivalence relations (SER)*, if* D *is a set and* E *is a set of equivalence relations on* D*. If* $d \in D$ *and* $\theta \in E$*, we denote the equivalence class of* d *by* $[d]\theta := \{d' \in D \mid d'\theta d\}$*. If* $\Delta_D \in E$ *we will also call* (D, E) *an "*SER *with identity relation".*

Every attribute $m \in M$ induces a partition on the object set via the equivalence classes of $\ker(m)$. So we can regard a simple many-valued context as a set of partitions induced by its attributes. Every block of a partition corresponds to the set of objects with a certain value with respect to a certain attribute. The following definition captures attribute-complete many-valued contexts.

Definition 5 (closed SER). *We call a pair (G, E) a* closed SER *if G is a set and E is a closure system of equivalence relations on G and E contains the identity on G.*

To every given closed SER $\mathbb{E} := (D, E)$ we can assign a simple many-valued context $\mathbf{K}(\mathbb{E}) := (D, E, W, I)$, where $W := \{[d]\theta \mid d \in D, \theta \in E\}$ and $I := \{(d, \theta, w) \in D \times E \times W \mid w = [d]\theta\}$. Obviously, $\mathbf{K}(\mathbb{E})$ is attribute-complete. On the other hand we can assign, as described above, a closed SER to every attribute-complete many-valued context. We define $\mathbf{E}(\mathbb{K}) := (G, \{\ker(m) \mid m \in M\})$. We observe that, for every attribute-complete many-valued context $\mathbb{K}$, we have $\mathbf{K}(\mathbf{E}(\mathbb{K})) \simeq \mathbb{K}$ and for every closed SER $\mathbb{E}$ we have $\mathbf{E}(\mathbf{K}(\mathbb{E})) = \mathbb{E}$.

If we have such a closed system of equivalence relations we can assign the lattice of its *labeled equivalence classes* to it. This structure, called *complete affine ordered set*, is axiomatized in the next chapter.

3 Complete Affine Ordered Sets

In [KS04] the labeled equivalence classes of a system of equivalence relations containing the diagonal are characterized order-theoretically using the notion of an *affine ordered set*. We recall this basic definition which we will specialize in the following yielding a corresponding notion to a closed SER.

Definition 6 (affine ordered set). *We call a triple $\mathbb{A} := (Q, \leq, \parallel)$ an* affine ordered set*, if $(Q, \leq)$ is a partially ordered set, $\parallel$ is a equivalence relation on Q, and the axioms* (A1) - (A4) *hold. Let $A(Q) := \mathrm{Min}(Q, \leq)$ denote the set of all minimal elements in $(Q, \leq)$ and $A(x) := \{a \in A(Q) \mid a \leq x\}$.*

(A1) $\forall x \in Q : A(x) \neq \emptyset$
(A2) $\forall x \in Q \forall a \in A \exists! t \in Q : a \leq t \parallel x$
(A3) $\forall x, y, x', y' \in Q : x' \parallel x \leq y \parallel y' \;\&\; A(x') \cap A(y') \neq \emptyset \Rightarrow x' \leq y'$
(A4) $\forall x, y \in Q \exists x', y' \in Q : x \not\leq y \;\&\; A(x) \subseteq A(y)$
$\Rightarrow x' \parallel x \;\&\; y' \parallel y \;\&\; A(x') \cap A(y') \neq \emptyset \;\&\; A(x') \nsubseteq A(y')$.

The elements of $A(Q)$ are called points *and, in general, elements of Q are called* subspaces. *We say that a subspace x is* contained *in a subspace y if $x \leq y$.*

For every $x \in Q$ we observe that $\theta(x) := \{(a, b) \in A^2 \mid \pi(a|x) = \pi(b|x)\}$ is an equivalence relation on the set of points.

Homomorphisms between ordered sets with parallelism are defined as follows:

Definition 7 (homomorphism for ordered sets with parallelism). *For ordered sets with parallelism $\mathbb{A} = (Q, \leq, \parallel)$ and $\mathbb{A}_0 = (Q_0, \leq_0, \parallel_0)$ we call a mapping $\alpha : Q \to Q_0$ a homomorphism if*

- *α maps points to points,*
- *α is order preserving (i.e. $x \leq y \implies \alpha(x) \leq_0 \alpha(y)$),*
- *α is preserves the parallelism (i.e. $x \parallel y \implies \alpha(x) \parallel_0 \alpha(y)$).*

By $\mathrm{Hom}(\mathbb{A}, \mathbb{A}_0)$ *we denote the set of all homomorphisms from* $\mathbb{A}$ *to* $\mathbb{A}_0$.

From [KS04] we know that we can assign to any affine ordered set $\mathbb{A}$ a SER with diagonal via $\mathbf{E}(\mathbb{A}) := (\mathrm{Min}(Q, \leq), \{\theta(x) | x \in Q\})$ and to any SER with diagonal $\mathbb{E}$ an affine ordered set via $\mathbf{A}(\mathbb{E}) := (\{([x]\theta, \theta) \mid \theta \in E\}, \leq', \|')$ where $\leq'$ is defined as $([x]\theta_1, \theta_1) \leq' ([y]\theta_2, \theta_2) :\iff [x]\theta_1 \subseteq [y]\theta_2 \;\;\&\;\; \theta_1 \subseteq \theta_2$ and $\|'$ is defined as $([x]\theta_1, \theta_1) \|' ([y]\theta_2, \theta_2) :\iff \theta_1 = \theta_2$.

For any ordered set P we denote by $P_\perp$ the order which results by adding a bottom element, also called the *lifting* of P. Given an affine ordered set $\mathbb{A} := (Q, \leq, \|)$, we will add as an axiom that $(Q, \leq)_\perp$ is a complete lattice to assure that the SER $\mathbf{E}(\mathbb{A})$ is closed, i.e. the set $\{\theta(x) \mid x \in Q\}$ forms a closure system.

Definition 8 (complete affine ordered set). *We call an affine ordered set* $\mathbb{C} := (Q, \leq, \|)$ complete affine ordered set *if* $(Q, \leq)_\perp$ *forms a complete lattice.*

As an illustration we give an example for a closed SER and its associated affine ordered set.

Example 1. We construct an affine ordered set from the following relations on a set $U := \{a, b, c, d, e\}$:

- $\triangle_U$
- θ_1 defined by the classes $\{a\}$ and $\{b, c, d, e\}$
- θ_2 defined by the classes $\{a, b\}$, $\{c\}$, and $\{d, e\}$
- θ_3 defined by the classes $\{b, c\}$ plus singletons
- θ_4 defined by the classes $\{d, e, \}$ plus singletons
- ∇_U

The lifting of the constructed affine ordered set is a lattice .

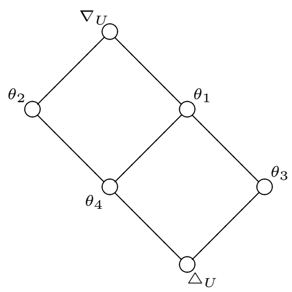

Fig. 1. Set of Equivalence Relations ordered via Set Inclusion

Note that for affine ordered sets we have $x \leq y \iff A(x) \subseteq A(y) \;\;\&\;\; \theta(x) \subseteq \theta(y)$.

Proposition 1. *Let* $\mathbb{A} := (Q, \leq, \|)$ *be an affine ordered set where* $(Q, \leq)_\perp$ *is a lattice and let* $x_i \in Q$ *for* $i \in I$. *Then we have* $A(\bigwedge_I x_i) = \bigcap_I A(x_i)$.

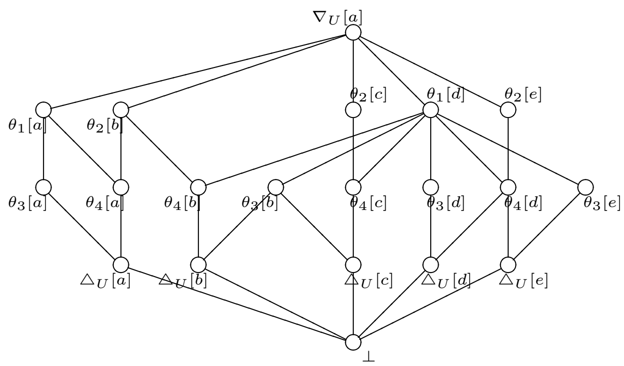

Fig. 2. Lifted Affine Ordered Set

Proof. Let $z := \bigwedge_I x_i$. We know that $A(z) \subseteq \bigcap_I A(x_i)$. Assume that there exists a $z^* \in \bigcap_I A(x_i)$ with $z^* \notin A(z)$. Assume that $A(\pi(z^*|z)) \subseteq \bigcap_I A(x_i)$. This contradicts the assumption that $(Q, \leq)$ is a lattice, since $\pi(z^*|z)$ would be a not comparable to z but also a lower bound of the x_i. So we have to assume $A(\pi(z^*|z)) \nsubseteq A(x) \cap A(y)$ which contradicts $\theta(z) \subseteq \bigcap_I \theta(x_i)$. □

Complete affine ordered sets exhibit a natural connection between parallelism and the meet of the lattice.

Proposition 2. *For a complete affine ordered set* $\mathbb{C} := (Q, \leq, \|)$ *we have*

(P1) $x_i \parallel y_i$ for all $i \in I$ & $\bigcap_{i\in I} A(x_i) \neq \emptyset$ & $\bigcap_{i\in I} A(y_i) \neq \emptyset$
$\implies \bigwedge_{i\in I} x_i \parallel \bigwedge_{i\in I} y_i$.

Proof. The premise yields elements $a, b \in A(Q)$ with $a \in \bigcap_{i\in I} A(x_i)$ and $b \in \bigcap_{i\in I} A(y_i)$. By (A3) we get $\pi(b|\bigwedge_{i\in I} x_i) \leq \bigwedge_{i\in I} y_i$ since $\pi(b|\bigwedge_{i\in I} x_i) \parallel \bigwedge_{i\in I} x_i \leq x_{i_0} \parallel y_{i_0}$ and $b \leq \pi(b|\bigwedge_{i\in I} x_i)$ and $b \leq y_{i_0}$. Exchanging the roles of the x_i and the y_i we dually get $\pi(a|\bigwedge_{i\in I} y_i) \leq \bigwedge_{i\in I} x_i$. But now assume $\pi(b|\bigwedge_{i\in I} x_i) < \bigwedge_{i\in I} y_i$. This would imply that $\bigwedge_{i\in I} y_i \nparallel \bigwedge_{i\in I} x_i$ and therefore we would get $\pi(a|\bigwedge_{i\in I} y_i) < \bigwedge_{i\in I} x_i$. But this yields a contradictory configuration as depicted in Figure 3. Therefore we have $\pi(b|\bigwedge_{i\in I} x_i) = \bigwedge_{i\in I} y_i$ which completes our proof.

□

4 The Correspondence between Closed SERs and Complete Affine Ordered Sets

In [KS04] it is shown that to any affine ordered set $\mathbb{A} = (Q, \leq, \|)$ the functor $\mathbf{E}$ assigns a SER with identity relation via $\mathbf{E}(\mathbb{A}) := (\mathrm{Min}(Q, \leq), \{\theta(x) | x \in Q\})$ to $\mathbb{A}$. Conversely, for a SER with identity relation $\mathbb{E} = (D, E)$ an affine ordered set $\mathbf{A}(\mathbb{E})$ can be constructed as follows:

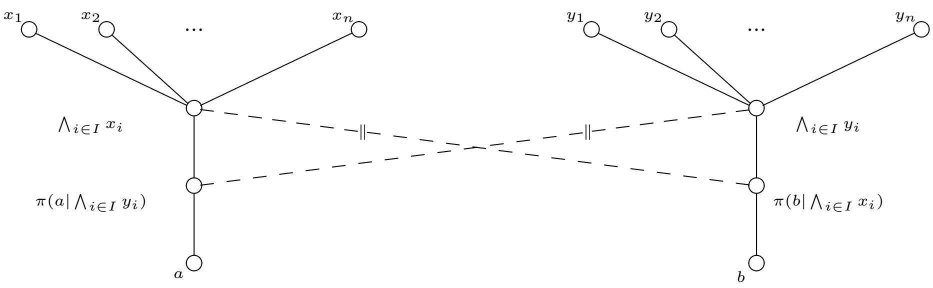

Fig. 3. Contradictory configuration for $I = \{1, ..., n\}$

- take the labeled equivalence classes $Q := \{([x]\theta, \theta) | x \in D\,,\, \theta \in E\}$ as set of subspaces of the affine ordered set
- define the order $\leq'$ on Q as $([x]\theta_1, \theta_1) \leq' ([y]\theta_2, \theta_2) :\iff [x]\theta_1 \subseteq [y]\theta_2$ & $\theta_1 \subseteq \theta_2$
- define a relation $\|'$ on the set of equivalence classes as $([x]\theta_1, \theta_1) \|' ([y]\theta_2, \theta_2) :\iff \theta_1 = \theta_2$

Theorem 2 in [KS04] includes the assertion that the functors **E** and **A** (extended to the respective homomorphisms) establish a categorical equivalence between SERs with diagonal and affine ordered sets. In the following we will show that these functors also yield a categorical equivalence between the category of closed SERs and the category of complete affine ordered sets.

Theorem 1. *The category of closed SERs and the category of complete affine ordered sets are equivalent.*

Proof. Since we know already that the category of affine ordered sets and the category of SER with identity are equivalent it remains to show that the functors **E** and **A** move complete affine ordered sets to closed SERs and vice versa. In the following let $\mathbb{C} := (Q, \leq, \|)$ be an complete affine ordered set. Firstly, we show that $\mathbf{E}(\mathbb{C}) = (E, D)$ is closed. Since we know that it contains the identity we have to show that D is a closure system. Let $R \subseteq Q$. Then we define $M_R := \{\theta(x) \,|\, x \in R\}$. We want to show that $\bigcap M_R \in \mathbf{E}(\mathbb{C})$. For this we construct an equivalence relation $\theta(z)$ and prove that $\theta(z) = \bigcap M_R$. Let $a \in A$ be an arbitrary but fixed point of $\mathbb{C}$. We define $z := \bigwedge_{x\in R} \pi(a|x)$. First, we show that $\theta(z) \subseteq \bigcap M_R$. For any $x \in R$ we have that $z \leq \pi(a|x)$. This implies that $\theta(z) \subseteq \theta(\pi(a|x)) = \theta(x)$. Second, we show that $\bigcap M_R \subseteq \theta(z)$. Let $(b, c) \in \theta(x)$ for all $x \in R$. Then we have $\pi(a|x) \parallel \pi(b|x) = \pi(c|x)$ for all $x \in R$. Since the $\bigcap_{x\in R} \pi(a|x) \supseteq \{a\}$ and $\bigcap_{x\in R} \pi(b|x) \supseteq \{b, c\}$ we can use (P1) to conclude that $z = \bigwedge_{x\in R} \pi(a|x) \parallel \bigwedge_{x\in R} \pi(b|x) = \bigwedge_{x\in R} \pi(a|c)$. But this shows that $(b, c) \in \theta(z)$.

For the other direction we have to show that $\mathbf{A}(\mathbb{E})$ is a complete affine ordered set for a closed SER $\mathbb{E} := (E, D)$. We know already that $\mathbf{A}(\mathbb{E})$ is an affine ordered set. So it remains to show that $\mathbf{A}(\mathbb{E})_\perp := (\{([x]\theta, \theta) \mid \theta \in D\}, \leq', \|')_\perp$ forms a complete lattice. We can consistently interpret $\perp$ as $(\emptyset, \emptyset)$. Let $S := \{([x]\theta, \theta) \mid \theta \in E\} \cup \{(\emptyset, \emptyset\}$ and let $R \subseteq S$. We define $\bigwedge R := (\bigcap \pi_1(R), \bigcap \pi_2(R))$. We have to show that $\bigwedge R \in S$. But since D is a closure system $\bigcap \pi_2(R) \in D$ having $\bigcap \pi_1(R)$ as a class. □

5 Coordinatization

In this section we give characterizations for the previously studied structures to be coordinatizable, that is, we characterize those structures whose carrier/point set consistently can be seen as the carrier set of an algebra.

5.1 Coordinatization of Closed SERs

At first, we investigate under which conditions a closed SER $\mathbb{E} = (A, D)$ can be coordinatized, that means, under which conditions there exists an algebra $\mathbb{A} := (A, (f)_I)$ with $\text{Con}(\mathbb{A}) = D$.

In the context of a SER $\mathbb{E}$ we define the set of *dilations* $\Delta(\mathbb{E})$ of the SER as all functions mapping points to points and respecting all equivalence relations in $\mathbb{E}$ (a map $\delta \in A^A$ respects an equivalence relation R on A if for all $(a, b) \in R$ we have $(\delta(a), \delta(b)) \in R$), that is, $\Delta(\mathbb{E}) := \{\delta \in A^A \mid \delta \text{ respects all } E \in D\}$. What makes dilations so interesting is the fact that congruence relations can already be characterized by their compatibility with unary polynomial functions. The set of all unary polynomial functions of an algebra $\mathbb{A}$ is denoted by $\Delta(\mathbb{A})$.

Proposition 3 ([Ih93], Theorem 1.4.8). *Let $\mathbb{A} := (A, (f)_I)$ be an algebra and $\theta \in EqA$. Then $\theta \in Con(\mathbb{A})$ if and only if θ respects all $\delta \in \Delta(\mathbb{A})$.*

Now it is easy to see that the dilations of a closed SER subsume the unary polynomial functions of a coordinatizing algebra if it exists.

Proposition 4. *Let $\mathbb{A} := (A, (f)_I)$ coordinatize $\mathbb{E} := (A, D)$. Then $\Delta(\mathbb{A}) \subseteq \Delta(\mathbb{E})$.*

Proof. We get a well known Galois connection between the set EqA of all equivalence relations on a set A and the set of all unary operations $Op_1(A)$ on that same set if we define the relation $I \subseteq EqA \times Op_1(A)$ via $EI\delta \iff \delta$ respects E. Since if $\mathbb{A} := (A, (f)_I)$ coordinatizes $\mathbb{E}$ by Proposition 3 we have $\Delta(\mathbb{A})^I = \text{Con}(\mathbb{A})$, and since $\cdot^{II}$ is a closure operator we get $\Delta(\mathbb{A}) \subseteq \Delta(\mathbb{A})^{II} = \Delta(\mathbb{E})$.

The following proposition gives an constructive view on the principal congruence relations of an algebra which will be useful in the proof of our characterization theorem for closed SERs.

Proposition 5. *Let* $\mathbb{A} := (A, (f)_I)$*, let* $\theta(a, b) \in \mathrm{Con}(\mathbb{A})$ *denote the least congruence relation* θ *with* $(a, b) \in \theta$*, and let* $\Delta(\mathbb{A})$ *denote the set of all unary polynomial functions of* $\mathbb{A}$*. Then* $\theta(a, b)$ *is the reflexive, symmetric, and transitive closure of*

$$\Delta(a, b) := \{(\delta a, \delta b) \mid \delta \in \Delta(\mathbb{A})\}.$$

The next theorem gives a characterization of coordinatizable closed SERs. Note that other characterizations of the set of congruence relations of an algebra are known, e.g. compare [Ih93], p. 56, Theorem 3.4.5. Before presenting our characterization which aims at providing some analogies to Theorem 3.5 in [Wi70] (where a geometry is coordinatized by the (not labeled) congruence classes of an algebra), we need one more definition:

Definition 9. *Let* A *be a set, let* $B \subseteq A$*, and let* Δ *be a set of maps from* A *to* A*. Then we define a relation* $\equiv \subseteq A \times A$ *such that for* $a, b \in A$ *we have* $a \equiv b \bmod (B, \Delta)$ *if and only if there exist* $\delta_i \in \Delta$ *for* $i = 0, ..., n$ *with* $a \in \delta_0(B)$ & $b \in \delta_n(B)$ & $\delta_i(B) \cap \delta_{i+1}(B) \neq \emptyset$ for $i \in \{1, ..., n-1\}$.

Theorem 2. *Let* $\mathbb{E} = (A, D)$ *be a closed SER. Then there exists an algebra* $\mathbb{A} := (A, (f)_I)$ *with* $\mathrm{Con}(\mathbb{A}) = D$ *if and only if*

(E1) $(c, d) \in \theta(a, b) \iff c \equiv d \bmod (\{a, b\}, \Delta(\mathbb{E}))$
(E2) $[(a, b) \in \theta \Rightarrow \theta_D(a, b) \subseteq \theta] \iff \theta \in D.$

A relation R *which fulfills the left hand side of the equivalence (E2) is called* one-closed *with respect to the closure operator* θ_D*. Then condition (E2) can be understood as saying that a one-closed relation is already closed.*

Proof. "$\Rightarrow$": Let $\mathbb{A} := (A, (f)_I)$ be an algebra with $\mathrm{Con}(\mathbb{A}) = D$ and let $(c, d) \in \theta(a, b)$. We will show that the conditions (E1) and (E2) hold. Since $\theta(a, b)$ is the least congruence relation in $\mathrm{Con}(\mathbb{A})$ containing (a, b), we know by Proposition 5 that $\theta(a, b)$ is the reflexive, transitive, and symmetric closure of $\Delta(a, b)$. Therefore there exist mappings $\delta_1, ..., \delta_n \in \Delta(\mathbb{A})$ with $\delta_0(a) = c$, $\delta_n(b) = d$, and $\delta_i \cap \delta_{i+1} \neq \emptyset$ and, using Proposition 4, we have $c \equiv d \bmod (\{a, b\}, \Delta(\mathbb{E}))$, which verifies condition (E1). To verify condition (E2), let θ be one-closed in $\mathrm{Con}(\mathbb{A})$. Now assume $\theta \notin \mathrm{Con}(\mathbb{A})$. Then there exist $(a_j, b_j) \in \theta$ for $j = 1, \ldots, n$ such that for some operation f of $\mathbb{A}$ with arity n we have $(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \notin \theta$. But let us consider the unary polynomial functions $\Gamma_i : A \to A$ for $i = 1, \ldots, n$ where $\Gamma_i(x) := f(b_1, \ldots, b_{i-1}, x, a_{i+1}, \ldots, a_n)$. We get

$$\begin{array}{lll} & \Gamma_1(a_1) & = f(a_1, a_2, a_3, \ldots, a_n) \\ \theta(a_1, b_1) & \Gamma_1(b_1) & = f(b_1, a_2, a_3, \ldots, a_n) \\ = & \Gamma_2(a_2) & = f(b_1, a_2, a_3, \ldots, a_n) \\ \theta(a_2, b_2) & \Gamma_2(b_2) & = f(b_1, b_2, a_3, \ldots, a_n) \\ & \vdots & \\ \theta(a_n, b_n) & \Gamma_n(b_n) & = f(b_1, \ldots, b_{n-1}, b_n). \end{array}$$

Since θ is one-closed, we receive $\theta(a_i, b_i) \subseteq \theta$ for $i = 1, \ldots, n$, and therefore, it follows that $(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in \theta$, a contradiction.

"$\Leftarrow$": Now let $\mathbb{E} = (A, D)$ be a closed SER satisfying condition (E1) and (E2). We will show that $D = \mathrm{Con}(\mathbb{A})$ for $\mathbb{A} := (A, \Delta(\mathbb{E}))$. By definition of $\Delta(\mathbb{E})$ all relations in D are congruence relations of the constructed algebra, that is, $D \subseteq \mathrm{Con}(\mathbb{A})$. It remains to show that D is "sufficiently large". For this we deduce that a congruence relation fulfills the left side of the equivalence (E2). Let $\theta \in \mathrm{Con}(\mathbb{A})$ and $(a, b) \in \theta$ and $(c, d) \in \theta_D(a, b)$. We have to show that $(c, d) \in \theta$. Since $\theta_D(a, b) \in D$ by (E1) we get $c \equiv d \bmod (\{a, b\}, \Delta(\mathbb{E}))$ which yields the existence of $\delta_i \in \Delta(\mathbb{E})$ for $i = 0, 1, \ldots, n$ with $a \in \delta_0(\{a, b\})$ & $b \in \delta_n(\{a, b\})$ & $\delta_i(\{a, b\}) \cap \delta_{i+1}(\{a, b\}) \neq \emptyset$ for $i \in \{1, 2, .., n-1\}$. Since θ is a congruence relation $(\delta_i(a), \delta_i(b)) \in \theta$. Transitivity yields $(\delta_0(a), \delta_n(b)) = (c, d) \in \theta$. This completes the proof.

□

5.2 Coordinatization of Complete Affine Ordered Sets

In the following we investigate under which conditions a complete affine ordered set $\mathbb{C}$ can be coordinatized, that means, under which conditions there exists an algebra $\mathbb{A}$ such that the elements of the complete affine ordered set can be interpreted as the labeled congruence classes of the algebra, precisely $\mathbb{C} \simeq \mathbf{A}(\mathrm{Con}(\mathbb{A}))$.

In the language of complete affine ordered sets the notion of a dilation reads as:

Definition 10. *Let $\mathbb{C} := (Q, \leq, \|)$ be a complete affine ordered set. Then we call a self map δ on $A(Q)$ a* dilation *if for all $a, b \in A(Q)$ it holds that $\delta(a) \leq \pi(\delta(b)|a \vee b)$. The set of all dilations of a complete affine ordered set is denoted by $\Delta(\mathbb{C})$.*

The dilations of a complete affine ordered set coincide with the dilations of its associated closed SER:

Proposition 6. *Let $\mathbb{C} := (Q, \leq, \|)$ be a complete affine ordered set. Then we have $\Delta(\mathbf{E}(\mathbb{C})) = \Delta(\mathbb{C})$.*

Proof. Let $a, b \in A(Q)$. We have $\delta(a) \leq \pi(\delta(b)|a \vee b)$ if and only if $(\delta(a), \delta(b)) \in \theta(a \vee b)$. □

Using Proposition 4 we see that the dilations of a complete affine ordered set also subsume the unary polynomial functions of a coordinatizing algebra if such an algebra exists.

For a complete affine ordered set $\mathbb{C}$, we call a partition $(C_i)_{i \in I}$ of the points of $\mathbb{C}$ *compatible* if for all $\delta \in \Delta(\mathbb{C})$ and for all $i \in I$ if $a, b \in C_i$ there exists a $j \in I$ such that $\delta(a), \delta(b) \in C_j$. Now we can state the coordinatization theorem for complete affine ordered sets as follows.

Theorem 3 (coordinatization of complete affine ordered sets). *Let $\mathbb{C} := (Q, \leq, \|)$ be a complete affine ordered set. Then $\mathbb{C}$ can be coordinatized if and only if*

(C1) *for any compatible partition* $(C_i)_{i\in I}$ *of* $A(Q)$ *there exist* $(x_i)_{i\in I}$ *with* $C_i = A(x_i)$ *for* $i \in I$ *and* $x_i \parallel x_j$ *for all* $i, j \in I$.

Proof. "⇒": Let $\mathbb{A} := (A, (f)_I)$ be an algebra that coordinatizes $\mathbb{C}$. To show (C1) let $(C_i)_{i\in I}$ be a compatible partition of $A(Q)$. By supposition $\mathbb{C}$ is isomorphic to $\mathbf{A}(\mathrm{Con}(\mathbb{A}))$. So we know there exists an isomorphism $\epsilon : Q \longrightarrow Q_{\mathbb{A}}$. The points of $\mathbf{A}(\mathrm{Con}(\mathbb{A}))$ are of the form $\{(a, \Delta_A), \mid a \in A\}$ and since we can identify points of $\mathbb{C}$ with points of $\mathbf{A}(\mathrm{Con}(\mathbb{A}))$ via ϵ we can also identify them with the carrier set A of $\mathbb{A}$. So we can recognize $\hat{C} := \bigcup_{i\in I} C_i^2$ as a congruence relation since the dilations of $\mathbb{C}$ subsume the unary polynomial functions of $\mathbb{A}$ and by Proposition 3 this is enough. But since $\hat{C}$ is a congruence relation we know that $(C_i, \hat{C})$ are mutually parallel subspaces of $\mathbf{A}(\mathrm{Con}(\mathbb{A}))$. And since $A(\epsilon^{-1}(C_i, \hat{C})) = \epsilon^{-1}(A(C_i, \hat{C})) = \epsilon^{-1}(\{(c, \Delta_A) \mid c \in C_i\}) = C_i$ we know that the required x_i exist and equal $\epsilon^{-1}(C_i, \hat{C})$.
"⇐": Let (C1) hold for a complete affine ordered set $\mathbb{C}$. In the following we will show that $\mathbb{A}_{\mathbb{C}} := (A(Q), \Delta(\mathbb{C}))$ coordinatizes $\mathbb{C}$. It suffices to prove that $\mathbf{E}(\mathbb{C}) = \mathrm{Con}(\mathbb{A}_{\mathbb{C}})$ since by Theorem 2 in [KS04] we know that $\mathbb{C} \simeq \mathbf{A}(\mathrm{Con}(\mathbb{A}_{\mathbb{C}}))$ is equivalent to $\mathbf{E}(\mathbb{C}) \simeq \mathbf{EA}(\mathrm{Con}(\mathbb{A}_{\mathbb{C}})) \simeq \mathrm{Con}(\mathbb{A}_{\mathbb{C}})$. Let $\theta \in \mathbf{E}(\mathbb{C})$. Obviously, for $(a, b) \in \theta$ we have that $(\delta(a), \delta(b)) \in \theta$ since δ is a dilation. Now assume that $\theta \in \mathrm{Con}(\mathbb{A}_{\mathbb{C}})$. Then $\{[a]\theta \mid a \in A\}$ constitutes a compatible partition of $\mathbb{C}$ since dilations respect the congruence relations of $\mathbb{A}_{\mathbb{C}}$ by construction. But this implies the existence of $(x_i)_{i\in I}$ with $x_i \parallel x_j$ and $A(x_i) = [a]\theta$ for some $a \in A$ and we have $\theta = \theta(x_i) \in \mathbf{E}(\mathbb{C})$ for an arbitrary $i \in I$. □

For a complete affine ordered set we can define a closure operator $\mathfrak{H}$ on the set of points $A(Q)$ via $\mathfrak{H}(P) := A(\bigvee P)$ for $P \subseteq A(Q)$. If a complete affine ordered set can be coordinatized this closure operator coincides with the closure operator assigning to each set of elements of an associated algebra the smallest congruence class they are contained in.

6 Outlook

As a *congruence class geometry* [Wi70] is a closure structure $(A, [\cdot])$ derived from an algebra $\mathbb{A} := (A, f_I)$ where $[\cdot]$ assigns to a set $C \subseteq A$ the smallest congruence class $[C]$ where C is contained in, it would be challenging to investigate the connection between complete affine ordered sets and congruence class geometries. Especially, this could enable utilitizations for data representations, since techniques for using congruence class geometries for data representation were insinuated in [Ka05]. For coordinatizable complete affine ordered sets, the structure $(A, \mathfrak{H})$ – where $\mathfrak{H}$ is defined as in the previous section – is already a congruence class geometry which "sits" in the affine ordered set. A first step could be to study the function which maps a coordinatizable affine ordered set onto the closed sets of a congruence class geometry of the respective algebra, denoted by $\mathrm{Kon}(\mathbb{A})$, via "forgetting the labels"; if we define $\mathrm{Kon}_l(\mathbb{A}) := \{(A, \theta) \in \mathrm{Kon}(\mathbb{A}) \times \mathrm{Con}(\mathbb{A}) \mid A \text{ is a class of } \theta\}$ this map is given by $\varphi : \mathrm{Kon}_l(\mathbb{A}) \to \mathrm{Kon}(\mathbb{A})$ with $\varphi((A, \theta)) := A$. The map φ is $\bigvee$-preserving and therefore residuated (if we attach bottom elements to source and target). Another viable extension of this line of research is constituted by the problem of coordinatizing *projective ordered sets*, the fourth categorical counterpart to simple many-valued contexts as formulated in [KS04].

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