Formal Topology, Chu Space and Approximable Concept

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Abstract. Within Martin-Lőf type theory ([4]), G. Sambin initiated the intuitionistic formal topology which includes Scott algebraic domain theory as a special case (unary formal topology)([7]). In [6], he introduced the notions of (algebraic) information base and translation, and proved the equivalence between the category of (algebraic) information bases and the category of (algebraic) Scott domains. In [1], B. Ganter, R. Wille initiated formal concept analysis, which is an order-theoretical analysis of scientific data. Concept is one of the main notions and tools. Zhang considered a special form of Chu space, and introduced the notion of approximable concept in [3, 9, 10], which is a generalization of concept. These are two "parallel worlds". In this paper, we introduce the notion of (new) information base, and investigate the relations between points of an information base and approximable concepts of a Chu space; the translations and context morphisms.

Keywords: information base, Chu space, approximable concept, translation, context morphism

1 Introduction

Within Martin-Lőf type theory [4], G. Sambin introduced formal topology, and proved that the category of unary formal topologies (information bases) with translations is equivalent to the category of Scott algebraic domains with Scott continuous mappings in [6]. In [7], he also introduced the new notion of formal topology.

In [1], B. Ganter, R. Wille initiated formal concept analysis, which is an order-theoretical analysis of scientific data. Concept is one of the main notions and tools. Zhang, P. Hitzler and Shen considered a special kind of Chu space, and introduced the notion of approximable concept, as a generalization of concept.

They obtained the equivalence between the category of formal contexts with context morphisms ([3]), the category of complete algebraic lattices with Scott

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continuous functions and the category of information systems (trivial consistency predicates) with approximable mappings.

Formal topology and formal concept analysis (Chu space, approximable concept) are two "parallel worlds". In this paper, we define a new notion of information base, and investigate the relation between them.

In this paper, we begin with an overview of information base, and Chu spaces, including Zhang's work, that is Section 2, surveys preliminaries. Then we investigate the relation between points of an information base and approximable concepts of a Chu space, that is Section 3. In the end, we investigate the relation between context morphisms and translations , i.e., Section 4.

2 Preliminaries

Let us recall some main notions needed in the paper. i.e., information base and Chu space. The other notions, for examples: algebraic lattice, Scott continuous mapping, Scott algebraic domain, etc., see [2, 9].

2.1 Information Base

Within Martin-Lőf type theory ([4]), G. Sambin initiated the intuitionistic formal topology which includes Scott algebraic domain theory as a special case (unary formal topology)([7]). In [6], he introduced the notions of (algebraic) information base and translation, and proved the equivalence between the category of (algebraic) information bases and the category of (algebraic) Scott domains, thus he obtained a new, simple representation of (algebraic) Scott domain. Information bases play the role which, in the customary approach, is played by two notions introduced by Scott, namely information systems and neighbourhood systems.

Information bases with translations form a category, and S. Valentini showed that it is cartesian closed in [8].

In [7], G. Sambin introduced the new notion of formal topology, corresponding to the new definition, we also obtain a new definition of information base.

Definition 1. An (algebraic) information base φ is a structure, i.e., $\varphi = \langle S, \cdot, Pos, \lhd \rangle$, where S is a set, $\cdot$ a binary associative operation called combination, Pos a property on S called positivity or consistency, and $\lhd$ a binary relation between elements of S called cover, which satisfy the following conditions, for $a, b, c \in S$.

(monotonicity) $$\frac{Pos(a) \quad a \lhd b}{Pos(b)}$$

(positivity) $$\frac{Pos(a) \to a \lhd b}{a \lhd b}$$

(reflexivity) $a \lhd a$

(transitivity) $\dfrac{a \lhd b \quad b \lhd c}{a \lhd c}$

(·−left) $\dfrac{a \lhd b}{a \cdot c \lhd b}$ and $\dfrac{a \lhd b}{c \cdot a \lhd b}$

(·−right) $\dfrac{a \lhd b \quad a \lhd c}{a \lhd b \cdot c}$

In fact, the definition of information base given by G. Sambin, there exists a distinguished element $\triangle$, called *unit*, and for every $a \in S$, $a \lhd \triangle$. In the above definition, we omit it. Definition 1 corresponds to the new definition of unary formal topology in [7].

As discussed in [6], an information base is a set S of pieces of information, $a \lhd b$ means a is more informative; a, b can always be put together in order to obtain a piece of information $a \cdot b$, which combines the information given by a and b; $Pos(a)$ implies that a is individually consistent, $Pos(a \cdot b)$ shows that a and b are compatible with the relation $\lhd$; and if a is more informative than b, the consistency of a implies that that of b. For more details, see [6].

The notion of a point of an information base was defined as follow.

Definition 2. A subset $\gamma \subseteq S$ is a point of an information base φ, if

1 (i) $\dfrac{a \in \gamma \quad b \in \gamma}{a \cdot b \in \gamma}$, (ii) $\dfrac{a \in \gamma \quad a \lhd b}{b \in \gamma}$,

2 $\dfrac{a \in \gamma}{Pos(a)}$.

A point is a filter of positive pieces of information. The set of all points of an information base φ, denoted by $Pt(\varphi)$.

In [6], G. Sambin introduced the notion of a translation F.

Definition 3. A relation F is called a translation between two information bases φ and $\phi = \langle T, \cdot, Pos, \lhd \rangle$, if for all $a, c \in S$ and $b, d \in T$:

1 (1) $\dfrac{aFb \quad aFd}{aFb \cdot d}$ (2) $\dfrac{aFb \quad b \lhd d}{aFd}$ (3) $\dfrac{Pos(a) \quad aFb}{Pos(b)}$

2 $\dfrac{a \lhd c \quad cFb}{aFb}$ 3 $\dfrac{Pos(a) \rightarrow aFb}{aFb}$.

2.2 Chu Space

As constructive models of linear logic, Barr and Seely brought Chu space to light in computer science. V. Pratt investigated the notion of Chu space in [5], and Zhang, P. Hitzler, Shen considered a special form of Chu spaces in [3, 9, 10] as follows.

Definition 4. A Chu space P is a triple $P = (P_o, \models_P, P_a)$, where P_o is a set of objects and P_a is a set of attributes. The satisfaction relation $\models_P$ is a subset of $P_o \times P_a$. A mapping from a Chu space $P = (P_o, \models_P, P_a)$ to a Chu space $Q = (Q_o, \models_Q, Q_a)$ is a pair of functions (f_a, f_o) with $f_a : P_a \to Q_a$ and $f_o : Q_o \to P_o$ such that for any $x \in P_a$ and $y \in Q_o$, $f_o(y) \models_P x$ iff $y \models_Q f_a(x)$.

With respect to a Chu space $P = (P_o, \models_P, P_a)$, two functions can be defined:

$\alpha : \mathcal{P}\ (P_o) \to \mathcal{P}\ (P_a)$ with $X \to \{a \mid \forall x \in X\ x \models_P a\}$,

$\omega : \mathcal{P}\ (P_a) \to \mathcal{P}\ (P_o)$ with $Y \to \{o \mid \forall y \in Y\ o \models_P y\}$.

α, ω form a Galois connection between $\mathcal{P}\ (P_o)$ and $\mathcal{P}\ (P_a)$, i.e., α, ω are anti-monotonic, and $\alpha \circ \omega$, $\omega \circ \alpha$ are idempotent.

Using the above two functions, Zhang and Shen introduced the notion of approximable concept in [9]. A subset $A \subseteq P_a$ is an approximable concept if for every finite subset $X \subseteq A$, we have $\alpha(\omega(X)) \subseteq A$.

3 Information Base and Chu Space

For any information base $\varphi = \langle S, \cdot, Pos, \lhd \rangle$. Let $Pos(S) = \{a \in S \mid Pos(a)\}$, and $\uparrow a = \{b \mid a \lhd b\}$.

Proposition 1. $P_\varphi = (Pt(\ \varphi)\ , \models_\varphi, Pos(S))$ is a Chu space, where

$\gamma \in Pt(\varphi),\ \ a \in Pos(S), \gamma \models_\varphi a$ iff $a \in \gamma$.

Proof. It is trivial.

Lemma 1. (1) $a \in S, Pos(a) \Rightarrow \uparrow a \in Pt(\varphi)$.

(2) $\gamma \in Pt(\varphi) \Rightarrow \gamma = \cup\{\uparrow a \mid a \in \gamma\}$.

Proof. It is trivial.

Lemma 2. For $a \in S, Pos(a) \Rightarrow \uparrow a$ is an approximable concept of P_φ.

Proof. For $\{b_1, b_2, \cdots, b_m\} \subseteq \uparrow a$.

$\omega(\{b_1, b_2, \cdots, b_m\}) = \{\beta \mid \beta \models_\varphi b_i, i = 1, 2, \cdots, m\} = \{\beta \mid b_i \in \beta, i = 1, 2, \cdots, m\}$.

This implies that $\uparrow (b_1 \cdot b_2 \cdot \cdots \cdot b_m) \in \omega(\{b_1, b_2, \cdots, b_m\})$.

$\forall x \in \alpha(\omega(\{b_1, b_2, \cdots, b_m\})) = \{x \mid x \models_\varphi \beta, \forall \beta \in \omega(\{b_1, b_2, \cdots, b_m\})\}$, that is to say, $x \in \beta$ for all $\beta \in \omega(\{b_1, b_2, \cdots, b_m\})$. Hence we have $x \in \uparrow (b_1 \cdot b_2 \cdot \cdots \cdot b_m)$, so $a \lhd (b_1 \cdot b_2 \cdot \cdots \cdot b_m) \lhd x$. By this , we get $x \in \uparrow a$, thus $\alpha(\omega(\{b_1, b_2, \cdots, b_m\})) \subseteq \uparrow a$, $\uparrow a$ is an approximable concept.

Lemma 3. Suppose $A \subseteq P_a\ (Pos(S))$ is an approximable concept, $a \in A \Rightarrow \uparrow a \subseteq A$.

Proof. By the definition of an approximable concept, we know that $\alpha(\omega(\{a\})) \subseteq A$.

Since $\omega(\{a\}) = \{\beta \mid \beta \models_\varphi a\} = \{\beta \mid a \in \beta\} = \{\beta \mid \uparrow a \subseteq \beta\}$, we have

$\alpha(\omega(\{a\})) = \{y \mid \forall \beta \in \omega(\{a\}), \beta \models_\varphi y\} = \{y \mid \forall \beta \in \omega(\{a\}), y \in \beta\}$.

So $\forall b \in \uparrow a$, we have $b \in \beta$ for all $\beta \in \omega(\{a\})$. This implies that $b \in \alpha(\omega(\{a\})$, thus $b \in A$.

By the above proof, we obtain that $\uparrow a \subseteq A$.

Proposition 2. $\gamma \subseteq S$ is a point of $\varphi \Leftrightarrow \gamma$ is a non-empty approximable concept of P_φ.

Proof. Suppose $\gamma \in Pt(\varphi)$, i.e., γ is a point. Since $\forall a \in \gamma, Pos(a)$, we have $\gamma \subseteq P_a$.

For any finite subset $\{a_1, a_2, \cdots, a_m\} \subseteq \gamma$.

$\omega(\{a_1, a_2, \cdots, a_m\}) = \{\beta \mid \forall i(i = 1, 2, \cdots, m), \beta \models_\varphi a_i\}$
$= \{\beta \mid \forall i(i = 1, 2, \cdots, m), a_i \in \beta\}$.

By Lemma 1, we have $\uparrow (a_1 \cdot a_2 \cdot \cdots \cdot a_m) \in \omega(\{a_1, a_2, \cdots, a_m\})$.

$\alpha(\omega(\{a_1, a_2, \cdots, a_m\})) = \alpha(\{\beta \mid \uparrow (a_1 \cdot a_2 \cdot \cdots \cdot a_m) \subseteq \beta\})$.

For $b \in \alpha(\omega\{a_1, a_2, \cdots, a_m\})$, and for every $\beta \in \{\beta \mid \uparrow (a_1 \cdot a_2 \cdot \cdots \cdot a_m) \subseteq \beta\}$, we have $\beta \models_\varphi b$.

This implies that $b \in \beta$ for all β of the above set. So $b \in \uparrow (a_1 \cdot a_2 \cdot \cdots \cdot a_m)$, thus $b \in \gamma$.

By the above proof, we obtain that γ is an approximable concept.

On the other hand, given an approximable concept $A \subseteq P_a(Pos(S))$ of the derived Chu space, we will prove that A is a point of the information base φ.

1(i) Assume that $x, y \in A$, by the definition of an approximable concept, we have $\alpha(\omega(\{x, y\})) \subseteq A$.

$\omega(\{x, y\}) = \{\beta \mid \beta \models_\varphi \{x, y\}\} = \{\beta \mid x \in \beta, y \in \beta\}$.

This implies that $x \cdot y \in \beta$ for all $\beta \in \omega(\{x, y\})$. So we obtain $x \cdot y \in \alpha(\omega(\{x, y\}))$, thus $x \cdot y \in A$.

1(ii) If $x \in A, x \lhd y$, by Lemma 3, $y \in A$.

2 $x \in A$, by the definition of P_a, we get $Pos(x)$.

By the above proof and Definition 3, we obtain that A is a point of the information base φ.

Conversely, suppose $P = (P_o, \models_P, P_a)$ is a Chu space. Let $S = Fin(P_a)$, the set of finite subsets of P_a. The elements of P_a will be noted by x, y, z; the subsets of P_a (the elements of S) denoted by u, v, w.

We define for every $u \in S, Pos(u)$; $u \cdot v = u \cup v$; $u \lhd v$ iff $v \subseteq \alpha(\omega(u))$.

Proposition 3. As defined above, $\varphi_P = (S, \cdot, Pos, \lhd)$ is an information base induced by a Chu space P.

Proof. By the above definition, we have to prove φ_P satisfies the transitivity property.

If $u \lhd v, v \lhd w$, then $v \subseteq \alpha(\omega(u))$, $w \subseteq \alpha(\omega(v))$.

$\omega(u) = \{o_u \mid o_u \models_P y, \forall y \in u\}$; $\omega(v) = \{o_v \mid o_v \models_P x, \forall x \in v\}$.

$\forall x \in v, x \in \alpha(\omega(u))$, we have for every $o_u \in \omega(u), o_u \models_P x$, thus $o_u \in \omega(v)$.

$\forall z \in w, z \in \alpha(\omega(v))$, we obtain that for every $o_v \in \omega(v), o_v \models_P z$, so $o_u \models_P z$.

This implies that $\forall o_u \in \omega(u), o_u \models_P z$. Hence $z \in \alpha(\omega(u)), w \subseteq \alpha(\omega(u))$, thus $u \lhd w$.

Lemma 4. Suppose $A \subseteq P_a$ is an approximable concept, then $\beta_A = \{u \mid u \in Fin(A)\}$ is a point of φ_P.

Proof. It is clear that β_A satisfies the conditions 1(i) and 2 of Definition 2. we have to prove that it satisfies 1(ii).

For $u \in \beta_A, v \in S, u \lhd v$, we have $v \subseteq \alpha(\omega(u)) \subseteq A$. But because v is a finite set, we get $v \in \beta_A$.

Lemma 5. Suppose $\beta \subseteq S$ is a point of φ_P, then $A_\beta = \cup\{\alpha(\omega(u)) \mid u \in \beta\}$ is an approximable concept.

Proof. For any subset $w = \{x_1, x_2, \cdots, x_m\} \subseteq A_\beta$, by the definition of A_β, there exist $u_1, u_2, \cdots, u_m \in \beta$, such that $x_i \in \alpha(\beta(u_i))$. Hence $u_i \lhd \{x_i\}$.

Since β is a point of φ_P, we have $u = u_1 \cup \cdots \cup u_m = u_1 \cdots \cdots u_m \lhd \{x_1\} \cdots \cdots \{x_m\} = \{x_1, \cdots, x_m\}$, and $u \in \beta$. By the definition of $\lhd$, $\{x_1, \cdots, x_m\} \subseteq \alpha(\omega(u))$. This implies that $\alpha(\omega(\{x_1, \cdots, x_m\})) \subseteq \alpha(\omega(\alpha(\omega(u)))) = \alpha(\omega(u)) \subseteq A_\beta$ ([2]). So A_β is an approximable concept.

Proposition 4. There exists a bijection between the set of points of φ_P and the set of approximable concepts of P.

Proof. (1) Given A is an approximable concept, by Lemma 4, we obtain a point $\beta_A = \{u \mid u \in Fin(A)\}$. By Lemma 5, we also know that $A_{\beta_A} = \cup\{\alpha(\omega(u)) \mid u \in \beta_A\}$ is an approximable concept of P. We try to prove $A = A_{\beta_A}$.

Clearly, $A \subseteq A_{\beta_A}$. For every $y \in A_{\beta_A}$, there exists $u \in \beta_A$, such that $y \in \alpha(\omega(u))$. Since $u \in \beta_A$, we have $u \in Fin(A)$, so $y \in \alpha(\omega(u)) \subseteq A$ by the definition of an approximable concept. Thus $A_{\beta_A} \subseteq A$.

(2) Given β is a point φ_P, then we obtain an approximable concept A_β by Lemma 5. But by Lemma 4, we also obtain a point β_{A_β} of φ_P. In the similar way, we may prove $\beta = \beta_{A_\beta}$.

4 Context Morphism and Translation

In [3], P.Hitzler and Zhang introduced the notion of a context morphism as follows.

Definition 5. Given formal contexts $P = (P_o, \models_P, P_a)$ and $Q = (Q_o, \models_Q, Q_a)$, a context morphism $\rightarrow_{PQ} = \rightarrow$ from P to Q is a relation $\rightarrow \subseteq Fin(P_a) \times Fin(Q_a)$, such that the following conditions are satisfied for all $X, X^{'}, Y_1, Y_2 \in Fin(P_a)$, and $Y, Y^{'} \in Fin(Q_a)$;

(1) $\emptyset \rightarrow \emptyset$,

(2) $X \rightarrow Y_1$ and $X \rightarrow Y_2$ implies $X \rightarrow Y_1 \cup Y_2$,

(3) $X^{'} \subseteq \alpha_P(\omega_P(X))$ and $X^{'} \rightarrow Y^{'}$ and $Y \subseteq \alpha_Q(\omega_Q(Y^{'}))$ imply $X \rightarrow Y$.

The category of formal contexts with context morphisms is cartesian closed ([3]).

Given a context morphism $\rightarrow_{PQ}$,we define a relation F^* between the derived information bases $\varphi_P = (S_P, \cdot, Pos, \lhd)$ and $\varphi_Q = (S_Q, \cdot, Pos, \lhd)$. For $u, v \in S_P, m, n \in S_Q$, uF^*v iff $u \rightarrow_{PQ} v$.

Lemma 6. F^* is a translation between φ_P and φ_Q.

Proof. By the definition of $\lhd$, the condition (3) in Definition 5 may be written as: $u \lhd v, vF^*m, m \lhd n$ imply uF^*n. By this, the proof is trivial.

For the other direction, suppose F is a translation between two information bases φ and ϕ, F determines a context morphism $\rightarrow_F$ from the Chu space $P_\varphi = (Pt(\varphi), \models_\varphi, Pos(S))$ to the Chu space $P_\phi = (Pt(\phi), \models_\phi, Pos(T))$. For $X \in Fin(Pos(S)), Y \in Fin(Pos(T))$, $X \rightarrow_F Y$ iff $\forall y \in Y,\ \exists x \in X,\ xFy$.

Lemma 7. $\rightarrow_F$ is a context morphism between P_φ and P_ϕ.

Proof. It is clear that to prove $\rightarrow_F$ satisfies the conditions (1) and (2) of Definition 5.

(3) If $X^{'} \subseteq \alpha_P(\omega_P(X))$, $X^{'} \rightarrow Y^{'}$ and $Y \subseteq \alpha_Q(\omega_Q(Y^{'}))$. Then for all $y \in Y$, $y^{'} \in Y^{'}$, we have $y^{'} \lhd y$.

Since $X^{'} \rightarrow Y^{'}$, for $y^{'} \in Y^{'}$, there exists $x^{'} \in X^{'}$, such that $x^{'}Fy^{'}$.

In the similarly way, for all $x \in X, x^{'} \in X^{'}$, we also have $x \lhd x^{'}$.

By the above proof and Definition 3, we obtain that xFy, so $X \to Y$.

By the above analysis, for an information base $\varphi = \langle S, \cdot, Pos, \lhd \rangle$, as defined in Proposition 1, we obtain a context $CT(\varphi) = P_{\varphi}$. On the other hand, given a Chu space P, we also get an information base $INB(P) = \varphi_P$, as defined in Proposition 3. Hence we have two functors CT and INB between the category of (new) information bases and the category of formal contexts.

We say two categories $\mathcal{C}$ and $\mathcal{D}$ is equivalent, if there exist functors $E : \mathcal{C} \to \mathcal{D}$ and $F : \mathcal{D} \to \mathcal{C}$, such that $E \circ F = id_{\mathcal{D}},\ F \circ E = id_{\mathcal{C}}$.

As showed above, G. Sambin introduced the new definition of formal topology in [7], and obtained that the category of (new) unary formal topologies is equivalent to the category of algebraic domains. The definition of (new) information base corresponds the new definition of unary formal topology, by this, we know that the category of (new) information bases is equivalent to the category of algebraic domains; by [9], we also know that the category of formal contexts is equivalent to the category of complete algebraic lattices; while the category of complete algebraic lattices is embedded into the category of algebraic domains, so the category of formal contexts is embedded into the category of information bases.

In [3], P. Hitzler investigated the category of information systems with trivial consistency predicate, i.e., Con = the set of all finite tokens. By this, we may define the subcategory of the category of information bases, where $\varphi = \langle S, \cdot, Pos, \lhd \rangle$, and $S = Pos$. Furthermore we may prove that the subcategory is equivalent to the category of formal contexts. So we obtain the following propositions.

Proposition 5. The following four categories are equivalent,

(1) the category of complete algebraic lattices and Scott continuous mappings,

(2) the category of formal contexts and context morphisms,

(3) the category of information systems with trivial consistency predicates and approximable mappings,

(4) the category of information bases with $S = Pos$ and translations.

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