Concept Lattices Constrained by Equivalence Relations

Radim Bělohlávek, Vladimír Sklenář, and Jiří Zacpal

Department of Computer Science, Palacky University, Olomouc
Tomkova 40, CZ-779 00 Olomouc, Czech Republic
radim.belohlavek@upol.cz

Abstract. Formal concept analysis is a method of exploratory data analysis that aims at the extraction of natural clusters from object-attribute data tables. The clusters, called formal concepts, are naturally interpreted as human-perceived concepts in a traditional sense and can be partially ordered by a subconcept-superconcept hierarchy. The hierarchical structure of formal concepts (so-called concept lattice) represents a structured information obtained automatically from the input data table.
This paper presents a preliminary study in which we deal with the problem of how further information additionally supplied with the basic object-attribute data table can be utilized. The additional information we consider has the form of a binary relation on the set of objects. Primarily, we focus on equivalence relations. Equivalences can be used modeling similarity, indistinguishability, etc.—a kind of information quite often supplied/available with a collection of objects. We aim at emphasizing two aspects. First, the additional information can provide a criterion for the relevance/importance of formal concepts. Only concepts which are in an appropriate sense compatible with the additional information are considered important. Second, selecting only important concepts means a reduction of the overall concept lattice which helps to make the resulting set of formal concepts more readable.

Keywords: formal concept analysis, concept lattice, constraint, binary relation, similarity

1 Introduction and problem setting

Patterns in data The search for interesting patterns in data has traditionally been a challenging problem. In the pre-computer era, the extent of efficiently analyzable data was small and the patterns looked for in the data were simple patterns easily recognizable and graspable by humans. Computers made it possible to analyze large amounts of data as well as to look for new kinds of patterns in the data. Formal concept analysis (FCA) [6] provides methods for finding patterns and dependencies in data which can be run automatically. The patterns looked for are called formal concepts. The attractiveness of formal concept analysis derives mainly from the fact that formal concepts are interpretable

© V. Snášel, R. Bělohlávek (Eds.): CLA 2004, pp. 58–66, ISBN 80-248-0597-9.
VŠB – Technical University of Ostrava, Dept. of Computer Science, 2004.

as natural concepts well-understood by humans. Both foundations and applications (classification, software (re)engineering, document and text organization, etc.) of formal concept analysis are well-documented (see [6] and [1], and the references therein).

Formal concept analysis In its basic setting, formal concept analysis deals with input data in the form of a table with rows corresponding to objects and columns corresponding to attributes which describes a relationship between the objects and attributes. The data table is formally represented by a so-called formal context which is a triplet $\langle X, Y, I\rangle$ where I is a binary relation between X and Y, $\langle x, y\rangle \in I$ meaning that the object x has the attribute y. For each $A \subseteq X$ denote by $A^{\uparrow}$ a subset of Y defined by

$$A^{\uparrow} = \{y \mid \text{for each } x \in X : \ \langle x, y\rangle \in I\}.$$

Similarly, for $B \subseteq Y$ denote by $B^{\downarrow}$ a subset of X defined by

$$B^{\downarrow} = \{x \mid \text{for each } y \in Y : \ \langle x, y\rangle \in I\}.$$

That is, $A^{\uparrow}$ is the set of all attributes from Y shared by all objects from A (and similarly for $B^{\downarrow}$). A formal concept in $\langle X, Y, I\rangle$ is a pair $\langle A, B\rangle$ of $A \subseteq X$ and $B \subseteq Y$ satisfying $A^{\uparrow} = B$ and $B^{\downarrow} = A$. That is, a formal concept consists of a set A (extent) of objects which fall under the concept and a set B (intent) of attributes which fall under the concept such that A is the set of all objects sharing all attributes from B and, conversely, B is the collection of all attributes from Y shared by all objects from A. The set $\mathcal{B}(X, Y, I) = \{\langle A, B\rangle \mid A^{\uparrow} = B, B^{\downarrow} = A\}$ of all formal concepts in $\langle X, Y, I\rangle$ can be naturally equipped with a partial order $\leq$ (modeling the subconcept-superconcept hierarchy, e.g. $\mathsf{dog} \leq \mathsf{mammal}$) defined by

$$\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle \text{ iff } A_1 \subseteq A_2 \text{ (or, equivalently, } B_2 \subseteq B_1).$$

Under $\leq$, $\mathcal{B}(X, Y, I)$ happens to be a complete lattice, called a concept lattice, the basic structure of which is described by the so-called main theorem of concept lattices [6, 9].

Theorem 1. *(1) The set $\mathcal{B}(X, Y, I)$ is under $\leq$ a complete lattice where the infima and suprema are given by*

$$\bigwedge_{j\in J} \langle A_j, B_j\rangle = \langle \bigcap_{j\in J} A_j, (\bigcup_{j\in J} B_j)^{\downarrow\uparrow}\rangle \ , \ \bigvee_{j\in J} \langle A_j, B_j\rangle = \langle (\bigcup_{j\in J} A_j)^{\uparrow\downarrow}, \bigcap_{j\in J} B_j\rangle \ . \quad (1)$$

(2) Moreover, an arbitrary complete lattice $\mathbf{V} = \langle V, \leq\rangle$ is isomorphic to $\mathcal{B}(X, Y, I)$ iff there are mappings $\gamma : X \to V$, $\mu : Y \to V$ such that

(i) $\gamma(X)$ is $\bigvee$-dense in V, $\mu(Y)$ is $\bigwedge$-dense in V;
(ii) $\gamma(x) \leq \mu(y)$ iff $\langle x, y\rangle \in I$.

Our aim Formal concept analysis thus treats both the individual objects and the individual attributes as distinct entities for which there is no further information available except for the relationship I saying which objects have which attributes. However, more often than not, both the set of objects and the set of attributes are supplied by an additional information. Further processing of the input data (formal context) should therefore take the additional information into account. Particularly, the conceptual clustering should take the additional information into account so that only those concepts which are in an appropriate sense compatible with the additional information are considered relevant. In this paper, we consider the additional information in the form of a binary relation on the set of objects. Our primary interest is in equivalence relations, describing similarity, insdistinguishability, etc., on objects. The main aim is to utilize the additional information to reduce the size of the resulting concept lattice. Section 2 provides a formal treatment of our approach, Section 3 presents illustrating examples, Section 4 gives some remarks on future research.

2 Concept lattices of contexts with binary relations

In what follows, we briefly present the conception and formal treatment of our approach.

Definition 1. *A formal context with a binary relation (R-context, for short) is a structure $\langle X, Y, I, \equiv\rangle$ (written also $\langle\langle X, \equiv\rangle, Y, I\rangle$) where $\langle X, Y, I\rangle$ is a formal context and $\equiv$ is a binary relation on X.*

Remark 1. (1) We are primarily interested in case when $\equiv$ is an equivalence relation. Then $x_1 \equiv x_2$ means that objects x_1 and x_2 are equivalent from some point of view (similar, indistuinguishable).

(2) Equivalence $\equiv$ may be supplied by an expert or may result from some previous analysis or external source. For example, objects from X may be partitioned by some clustering (based on attributes from Y or some other data available) or some convention (a catalogue). Such a partition gives naturally a rise to an equivalence relation.

If $\equiv$ represents an indistinguishability (or intended indistinguishability), it might be desirable to consider only those formal concepts which do not separate indistinguishable objects. We call such formal concepts compatible.

Definition 2. *For an R-context $\langle\langle X, \equiv\rangle, Y, I\rangle$, a formal concept $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$ is called compatible with $\equiv$ if for each $x_1, x_2 \in X$, if $x_1 \in A$, and $x_1 \equiv x_2$ or $x_2 \equiv x_1$, then $x_2 \in A$.*

Compatible concepts are thus certain formal concepts from $\mathcal{B}(X, Y, I)$ satisfying a natural restriction with respect to $\equiv$. The set of all formal concepts from $\mathcal{B}(X, Y, I)$ which are compatible with $\equiv$ will be denoted by $\mathcal{B}(\langle X, \equiv\rangle, Y, I)$, i.e.

$$\mathcal{B}(\langle X, \equiv\rangle, Y, I) = \{\langle A, B\rangle \in \mathcal{B}(X, Y, I) \mid \text{for each } x_1, x_2 :\ x_1 \in A,\ x_1 \equiv x_2 \text{ or } x_2 \equiv x_1 \text{ implies } x_2 \in A\}.$$

The following lemma is obvious.

Lemma 1. *If $\equiv$ is symmetric then $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$ is compatible with $\equiv$ iff for each $x_1, x_2 \in X$, if $x_1 \in A$ and $x_1 \equiv x_2$ then $x_2 \in A$*

For an equivalence $\equiv$ on X, compatible formal concepts are unions of $\equiv$-classes (recall that an $\equiv$-class corresponding to $x \in X$ is a set $[x]_\equiv = \{x' \in X \mid x \equiv x'\}$; the collection of all $\equiv$-classes is denoted by $X/\equiv$).

Theorem 2. *A formal concept $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$ is compatible with $\equiv$ iff A is a union of some $\equiv$-classes, i.e. there is $\mathcal{A} \subseteq X/\equiv$ such that $A = \bigcup \mathcal{A}$.*

Proof. The proof is almost evident (it follows from Definition 2 and the definition of an equivalence class). □

It is obvious that $\mathcal{B}(\langle X, \mathrm{id}_X\rangle, Y, I) = \mathcal{B}(X, Y, I)$, i.e. if $\equiv$ is the identity on X then any formal concept of $\mathcal{B}(X, Y, I)$ is compatible with $\equiv$ (this agrees with the intended way of restriction by $\equiv$). The same holds true for $\equiv = \emptyset$ (the restriction formulated by $\equiv$ is empty), i.e. $\mathcal{B}(\langle X, \emptyset\rangle, Y, I) = \mathcal{B}(X, Y, I)$. More generally, we can proceed as follows. For a formal context $\langle X, Y, I\rangle$ denote by $\cong_X$ the binary relation defined on X by

$$x_1 \cong_X x_2 \quad \text{if and only if for each } y \in Y:\ \langle x_1, y\rangle \in I \text{ iff } \langle x_2, y\rangle \in I. \tag{2}$$

In other words, $x_1 \cong_X x_2$ if and only if x_1 and x_2 have the same set of attributes, i.e. if $\langle \{x_1\}^{\uparrow\downarrow}, \{x_1\}^{\uparrow}\rangle = \langle \{x_2\}^{\uparrow\downarrow}, \{x_2\}^{\uparrow}\rangle$. Obviously, $\cong_X$ is an equivalence relation on X. We have the following statement.

Theorem 3. *$\mathcal{B}(\langle X, \equiv\rangle, Y, I) = \mathcal{B}(X, Y, I)$ if and only if for each $x_1, x_2 \in X$, $x_1 \equiv x_2$ implies $x_1 \cong_X x_2$.*

Proof. We omit the proof (due to the limited scope). □

Corollary 1. *$\mathcal{B}(\langle X, \equiv\rangle, Y, I) = \mathcal{B}(X, Y, I)$ if and only if for each $x_1, x_2 \in X$, if x_1 and x_2 are separated by some $y \in Y$, then $x_1 \not\equiv x_2$.*

The next theorem shows a natural result saying that the more restrictions, the less formal concepts satisfying the restrictions.

Theorem 4. *If $\equiv_1 \subseteq \equiv_2$ then $\mathcal{B}(\langle X, \equiv_2\rangle, Y, I) \subseteq \mathcal{B}(\langle X, \equiv_1\rangle, Y, I)$.*

Proof. Trivial. □

Given a formal context $\langle X, Y, I\rangle$ and a binary relation $\equiv$ on X, a natural question arises for what binary relations Q on X we have $\mathcal{B}(\langle X, \equiv\rangle, Y, I) = \mathcal{B}(\langle X, Q\rangle, Y, I)$, i.e. what Q are restrictive to the same extent as $\equiv$. We will answer the question with respect to the operations of equivalential closure. For a binary relation R, the equivalential closure will be denoted by R^{E}. By definition, R^{E} is the least equivalence relation containing R (i.e. $R \subseteq R^{\mathrm{E}}$).

Theorem 5. *For an R-context $\langle\langle X, \equiv\rangle, Y, I\rangle$ we have*

$$\mathcal{B}(\langle X, \mathrm{red}(\equiv)\rangle, Y, I) = \mathcal{B}(\langle X, \equiv\rangle, Y, I) = \mathcal{B}\left(\langle X, \equiv^{E}\rangle, Y, I\right)$$

where $\mathrm{red}(\equiv)$ *is any relation such that* $\mathrm{red}(\equiv) \subseteq \equiv - \mathrm{id}_X - \equiv^2$ *and for each* $x_1 \mathrm{red}(\equiv) x_2$ *we have* $x_2 \neg \mathrm{red}(\equiv) x_1$ *(i.e.* $\mathrm{red}(\equiv)$ *is an equivalential reduction of* $\equiv$*). Furthermore, for each binary relation Q on X satisfying* $\mathrm{red}(\equiv) \subseteq Q \subseteq \equiv^{E}$

we have $\mathcal{B}(\langle X, Q\rangle, Y, I) = \mathcal{B}(\langle X, \equiv\rangle, Y, I)$.
Proof. We omit the proof (due to the limited scope).
□

Theorem 5 shows natural bounds (in terms of transitive reduction and closure) on relation Q which are equally restrictive as $\equiv$.

The restriction of the subconcept-superconcept hierarchy $\leq$ which is defined on $\mathcal{B}(X, Y, I)$ makes $\langle\langle X, \equiv\rangle, Y, I\rangle$ itself a partially ordered set $\langle\mathcal{B}(\langle X, \equiv\rangle, Y, I), \leq\rangle$. The following theorem shows that $\langle\langle X, \equiv\rangle, Y, I\rangle$ is itself a complete lattice which is a reasonable substructure of the whole concept lattice $\mathcal{B}(X, Y, I)$.

Theorem 6. $\mathcal{B}(\langle X, \equiv\rangle, Y, I)$ *equipped with* $\leq$ *is a complete lattice in which arbitrary infima coincide with infima in* $\mathcal{B}(X, Y, I)$*, i.e. it is a complete* $\bigwedge$*-sublattice of* $\mathcal{B}(X, Y, I)$.
Proof. It can be shown (we omit details) that for any $\langle A_i, B_i\rangle \in \mathcal{B}(\langle X, \equiv\rangle, Y, I)$ we have $\langle \bigcap_i A_i, (\bigcap_i A_i)^{\uparrow}\rangle \in \mathcal{B}(\langle X, \equiv\rangle, Y, I)$. That is, $\mathcal{B}(\langle X, \equiv\rangle, Y, I)$ is closed under arbitrary infima from $\mathcal{B}(X, Y, I)$ which gives the claim. □

It can be shown by an easy example that suprema in $\mathcal{B}(\langle X, \equiv\rangle, Y, I)$ do not generally coincide with suprema in $\mathcal{B}(X, Y, I)$.

3 Examples and discussion

We now present illustrative examples. We assume that the reader is familiar with Hasse diagrams which will be used for visualization of concept lattices and attribute hierarchies. We label the nodes corresponding to formal concepts by boxes containing concept descriptions. For example, $(\{1, 3, 7\}, \{3, 4\})$ is a description of a concept the extent of which consists of objects 1, 3, and 7, and the intent of which consists of attributes 3 and 4.

Example 1. The following example shows the effect of reduction of the number of formal concepts. Consider the formal context $\langle X, Y, I\rangle$ in Tab. 1 .

The concept lattice $\mathcal{B}(X, Y, I)$ corresponding to formal context $\langle X, Y, I\rangle$ contains 19 formal concepts and is depicted in Fig. 1. Consider furthermore the equivalence relation $\equiv$ given by the type of fund. There are four $\equiv$-classes corresponding to stock funds, bond funds, mixed funds and money funds. The set of all formal concepts from $\mathcal{B}(X, Y, I)$ which are compatible with $\equiv$ contains 6 formal concepts and is depicted in Fig. 2.

Example 2. The following example demonstrates that the restriction can be quite extensive. Consider the formal context $\langle X, Y, I\rangle$ in Tab. 2 . The concept lattice $\mathcal{B}(X, Y, I)$ corresponding to formal context $\langle X, Y, I\rangle$ contains 21 formal concepts and is depicted in Fig. 3. Consider furthermore an equivalence $\equiv$ induced in salary so that we have 3 classes corresponding to salary less then 13 000, salary between 13 000 and 17 000 and salary higher then 17 000. The set of all formal concepts from $\mathcal{B}(X, Y, I)$ which are compatible with $\equiv$ contains 3 formal concepts and is depicted in Fig. 4.

Table 1. Formal context from Example 1

	fund	type	1 2 3	4 5 6	7 8 9
1	CPI Penezniho trhu	money	1 0 0	1 0 0	0 1 0
2	CSOB Akciovy	stock	1 0 0	0 0 1	0 0 1
3	CSOB Bond mix	bond	0 1 0	1 0 0	0 1 0
4	IKS Dluhopisovy	bond	0 1 0	1 0 0	1 0 0
5	IKS Globalni	mixed	0 1 0	0 1 0	0 1 0
6	IKS Penezni trh	money	1 0 0	0 1 0	0 1 0
7	ISCS Sporoinvest	money	1 0 0	0 1 0	0 1 0
8	ISCS Sporotrend	stock	0 0 1	0 0 1	0 0 1
9	ISCS Trendbond	bond	0 0 1	1 0 0	1 0 0
10	ISCS Vynosovy	mixed	0 0 1	0 1 0	0 1 0

attributes: 1 - rating for 1 week $<= 0,5$, 2 - rating for 1 week $> 0,5$ and $<= 1$, 3 - rating for 1 week > 1, 4 - rating for 26 weeks $<= 0,5$, 5 - rating for 26 weeks $> 0,5$ and $<= 4$, 6 - rating for 26 weeks > 4, 7 - rating for 52 weeks $<= 0,5$, 8 - rating for 56 weeks $> 0,5$ and $<= 10$, 9 - rating for 56 weeks > 10

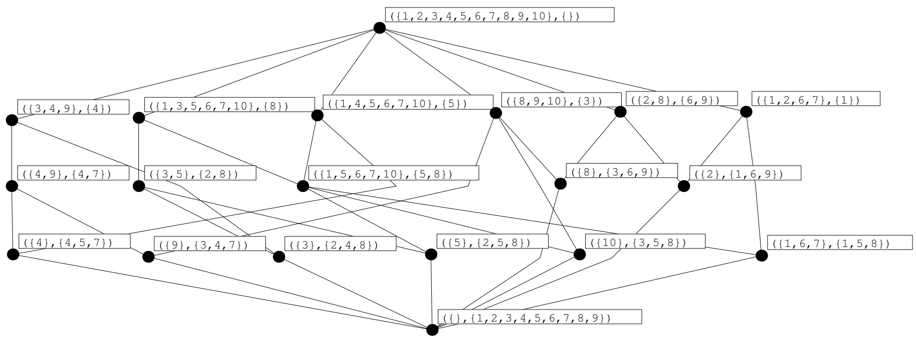

Fig. 1. Concept lattice corresponding to the context from Tab. 1

Table 2. Formal context from Example 2

	name	salary	1 2 3 4 5 6 7
1	Iva	15 000	1 0 0 1 1 0 0
2	Iva	15 000	0 0 0 1 1 0 0
3	Jirka	19 000	1 1 1 0 1 0 0
4	Jitka	14 000	1 0 1 0 0 1 1
5	Karel	15 000	1 0 0 1 0 1 1
6	Marek	13 000	0 0 1 0 1 0 0
7	Pavel	25 000	1 1 1 0 1 0 1
8	Pavla	14 000	1 0 0 1 1 0 0
9	Petr	20 000	1 1 0 1 1 0 1
10	Radim	13 000	1 0 1 0 0 1 0

attributes: 1 - secondary school (education), 2 - university (education), 3 - single, 4 - married, 5 - lives in a flat, 6 - owns a house, 7 - owns a car

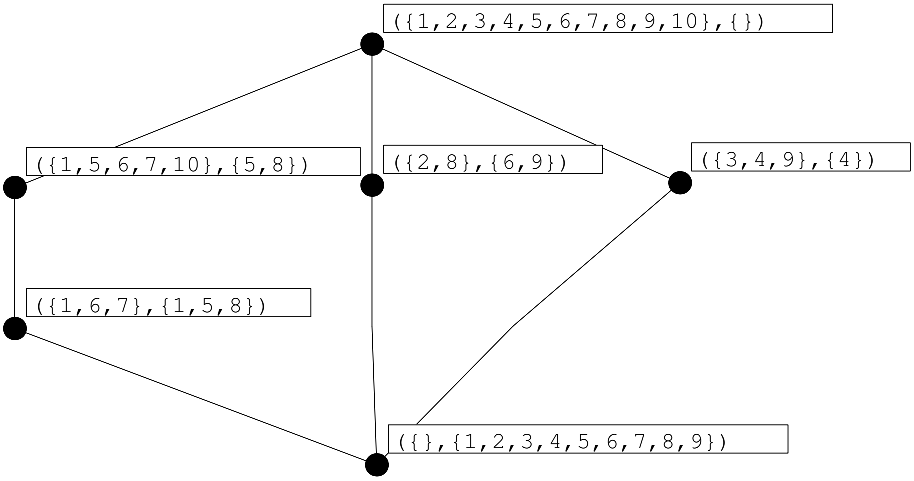

Fig. 2. Concepts corresponding to the context from Tab. 1 which are compatible with ≡ from Example 1

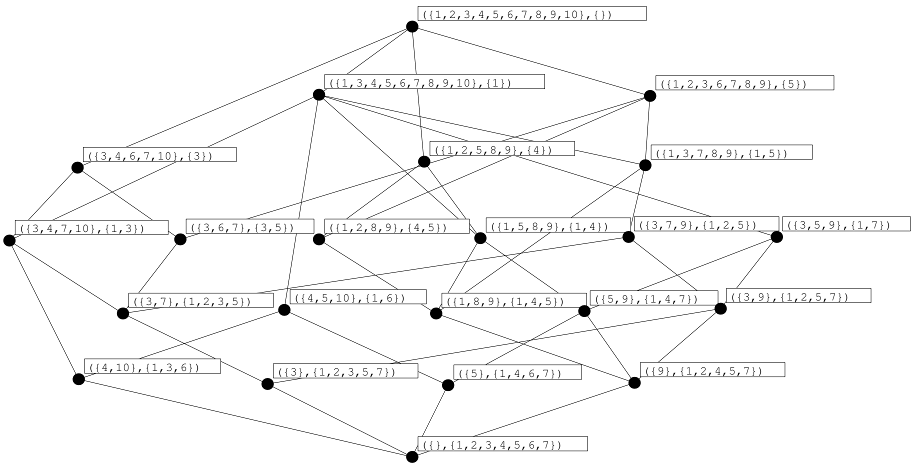

Fig. 3. Concept lattice corresponding to the context from Tab. 2

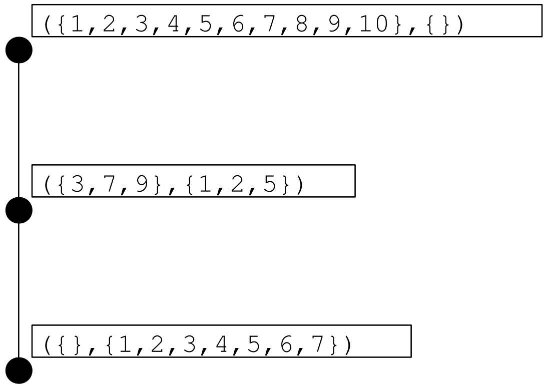

Fig. 4. Concepts corresponding to the context from Tab. 2 which compatible with $\equiv$ from Example 2

4 Future research

We now comment on some further topics and future research (some of these are studied in [4]).

- Relation to rough sets. Having an equivalence relation on the universe set and considering only sets which are using our terminology compatible with the equivalence is the main idea of so-called rough sets introduced by Pawlak [8]. This gave rise to so-called rough concept analysis, see e.g. [7], on which we recently learned. An investigation of the relationship with our approach is in progress (note that the notion of a definable formal concept as defined in [7] coincides with our notion of a compatible formal concept; there is, however, almost no overlap between [7] and Section 2).
- There is another interesting way to formulate a constraint. Namely, requiring that for a formal concept $\langle A, B\rangle$, all objects from A are $\equiv$-equivalent. We elaborate more on this in [4].
- A concept lattice may be thought of as a hierarchical clustering scheme. The partition corresponding to $\equiv$ represents another clustering (more generally, we can think of a hierarchical clustering scheme). Several interesting problems arise here (constraining one clustering by the other, comparing the clusterings, measuring their mutual consistency, etc.), a work is in progress.

Acknowledgement The research of the first author was supported by grant No. 201/02/P076 of the GAČR.

References

1. http://www.mathematik.tu-darmstadt.de/ags/ag1/Literatur/literatur_de.html

2. Arnauld A., Nicole P.: *La logique ou l'art de penser.* 1662. Also in German: *Die Logik oder die Kunst des Denkens.* Darmstadt, 1972.
3. Bělohlávek R., Sklenář V., Zacpal J.: Formal concept analysis with hierarchically ordered attributes. *Int. J. General Sytems* (to appear).
4. Bělohlávek R., Sklenář V., Zacpal J.: Formal concept analysis constrained by partitions (in preparation).
5. Birkhoff G.: *Lattice Theory, 3-rd edition.* AMS Coll. Publ. 25, Providence, R.I., 1967.
6. Ganter B., Wille R.: *Formal concept analysis. Mathematical Foundations.* Springer-Verlag, Berlin, 1999.
7. Kent R. E.: Rough concept analysis. In: Ziarko W. P. (Ed.): *Rough Sets, Fuzzy Sets, and Knowledge Discovery.* Proc. of the Intern. Workshop RSKD'93, Springer-Verlag, London, 1994.
8. Pawlak Z.: *Rough Sets: Theortical Aspects of Reasoning About Data.* Kluwer, Dordrecht, 1992.
9. Wille R.: Restructuring lattice theory: an approach based on hierarchies of concepts. In: Rival I.: *Ordered Sets.* Reidel, Dordrecht, Boston, 1982, 445—470.