The Basic Theorem on Generalized Concept Lattice

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Abstract. In [4] we have presented a new common platform for different types of fuzzification of a concept lattice. Now we show a pendant of the basic theorem on clasical concept lattices for this generalization which characterizes (complete) lattices isomorphic to this generalized concept lattice.

1 Introduction

There are some attempts have arisen which try to fuzzify the classical crisp Ganter-Wille's concept lattice, i.e. to consider the fuzzy values instead of 0's/1's in a matrix. One of them is the natural (and symmetric) fuzzification given by e.g. Bělohlávek ([1]) which consider (L-)fuzzy subsets of objects and (L-)fuzzy subsets of attributes. Another from author's paper [4] is a "bad boy", because it is not symmetric: it considers fuzzy subsets of attributes but ordinary/classical/crisp subsets of objects.

Our paper [4] shows a relationship between these approaches by finding certain common platform for both of them. The main idea of it is to differ between types of the fuzziness of objects and of the fuzziness of attributes as it is used in its special case in [3].

In this paper we formulate the extend version of the basic theorem on such generalized concept lattice which is the generalization of above-mentioned and we proved this added extension (the simple version of it was proved in [4]). This extension characterizes all lattices which are isomorphic to a generalized concept lattice.

2 A generalized fuzzy concept lattice

Now we shortly repeat basic definitions and results from [4]:

Let L be a poset, C and D be supremum-complete upper-semilattices (i.e. there exists $\sup X = \bigvee X$ for each subset X of C or D). Let $\bullet : C \times D \to L$ be monotone and left-continuous in both their arguments, i.e.

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1a) $c_1 \leq c_2$ implies $c_1 \bullet d \leq c_2 \bullet d$ for all $c_1, c_2 \in C$ and $d \in D$.
1b) $d_1 \leq d_2$ implies $c \bullet d_1 \leq c \bullet d_2$ for all $c \in C$ and $d_1, d_2 \in D$.
2a) If $c \bullet d \leq \ell$ holds for $d \in D$, $\ell \in L$ and for all $c \in X \subseteq C$, then $\sup X \bullet d \leq \ell$.
2b) If $c \bullet d \leq \ell$ holds for $c \in C$, $\ell \in L$ and for all $d \in Y \subseteq D$, then $c \bullet \sup Y \leq \ell$.

Let A and B be non-empty sets and let R be L-fuzzy relation on their Cartesian product, i.e. $R : A \times B \to L$.

Define the following mapping $\nearrow : {}^B D \to {}^A C$:

If $g : B \to D$ then $\nearrow(g) : A \to C$ is defined in the following way:

$$\nearrow(g)(a) = \sup\{c \in C : (\forall b \in B) c \bullet g(b) \leq R(a, b)\}.$$

Symmetrically we define the mapping $\swarrow : {}^A C \to {}^B D$:

If $f : A \to C$ then $\swarrow(f) : B \to D$ is defined in the following way:

$$\swarrow(f)(b) = \sup\{d \in D : (\forall a \in A) f(a) \bullet d \leq R(a, b)\}.$$

We have shown in [4] that these two mappings form a Galois connection and they are generalization of classical crisp Ganter-Wille's approach (see [2]), fuzzy Bělohlávek's (or independently Pollandt's) one (see [1], [5]) and our one-sided fuzzy one (see [3]).

Hence we use our new mappings $\nearrow$ and $\swarrow$ for constructing a concept lattice following the original Ganter-Wille's method from [2].

We assume from now that C and D are complete lattices. A *concept* is a pair $\langle g, f \rangle \in {}^B D \times {}^A C$ s.t. $\nearrow(g) = f$ and $\swarrow(f) = g$. If $\langle g_1, f_1 \rangle$ and $\langle g_2, f_2 \rangle$ are concepts, we will write $\langle g_1, f_1 \rangle \leq \langle g_2, f_2 \rangle$ iff $g_1 \leq g_2$ (or equivalently $f_1 \geq f_2$). The set of all such concepts with the order $\leq$ is called a *(generalized) concept lattice* and denoted shortly by $\mathfrak{L}$.

Theorem 1 (The Basic Theorem on Generalized Concept Lattice).

1) The generalized concept lattice $\mathfrak{L}$ is a complete lattice in which

$$\bigwedge_{i \in I} \langle g_i, f_i \rangle = \left\langle \bigwedge_{i \in I} g_i, \nearrow\left(\swarrow\left(\bigvee_{i \in I} f_i\right)\right)\right\rangle$$

and

$$\bigvee_{i \in I} \langle g_i, f_i \rangle = \left\langle \swarrow\left(\nearrow\left(\bigvee_{i \in I} g_i\right)\right), \bigwedge_{i \in I} f_i \right\rangle.$$

2) Let moreover L have the least element 0_L and $0_C \bullet d = 0_L$ and $c \bullet 0_D = 0_L$ for every $c \in C$ and $d \in D$. Then a complete lattice V is isomorphic to $\mathfrak{L}$ if and only if there are mappings $\alpha : A \times C \to V$ and $\beta : B \times D \to V$ s.t.
1a) α is non-increasing in the second argument.
1b) β is non-decreasing in the second argument.
2a) $\alpha[A \times C]$ is infimum-dense.
2b) $\beta[B \times D]$ is supremum-dense.

3) For every $a \in A$, $b \in B$, $c \in C$, $d \in D$

$$\alpha(a,c) \geq \beta(b,d) \qquad \text{if and only if} \qquad c \bullet d \leq R(a,b).$$

The proof of the part 1) is in [4], so we prove the second part now.

We will need the following *singleton functions*: If T is an arbitrary set and U is a poset with the least element 0_U, define $\mathrm{S}^{T,U}_{t,u} : T \to U$ in the following way:

$$\mathrm{S}^{T,U}_{t,u}(x) = \begin{cases} u, & \text{if } x = t, \\ 0_U, & \text{elsewhere.} \end{cases}$$

We show a few basic properties of these functions.

Lemma 1.

$$f = \sup\{\mathrm{S}^{T,U}_{t,f(t)} : t \in T\}.$$

Proof. Let $x \in T$. Because of pointwise defined supremum of a set of functions we have $\sup\{\mathrm{S}^{T,U}_{t,f(t)} : t \in T\}(x) = \sup\{\mathrm{S}^{T,U}_{t,f(t)}(x) : t \in T\}$. For all $t \neq x$ we have $\mathrm{S}^{T,U}_{t,f(t)}(x) = 0_U$, hence $\sup\{\mathrm{S}^{T,U}_{t,f(t)}(x) : t \in T\} = \mathrm{S}^{T,U}_{x,f(x)}(x) = f(x)$. So we obtain $f(x) = \sup\{\mathrm{S}^{T,U}_{t,f(t)} : t \in T\}(x)$ for all $x \in T$, q.e.d..

Lemma 2. *a) For all $a \in A$, $b \in B$, $c \in C$*

$$\swarrow(\mathrm{S}^{A,C}_{a,c})(b) = \sup\{d \in D : c \bullet d \leq R(a,b)\}.$$

b) For all $a \in A$, $b \in B$, $d \in D$

$$\nearrow(\mathrm{S}^{B,D}_{b,d})(a) = \sup\{c \in C : c \bullet d \leq R(a,b)\}.$$

Proof. By the definition $\swarrow(\mathrm{S}^{A,C}_{a,c})(b) = \sup\{d \in D : (\forall x \in A)\mathrm{S}^{A,C}_{a,c}(x) \bullet d \leq R(x,b)\}$. Because $\mathrm{S}^{A,C}_{a,c}(x) \bullet d = 0_C \bullet d = 0_L$ for every $x \neq a$, we obtain $(\forall x \in A)\mathrm{S}^{A,C}_{a,c}(x) \bullet d \leq R(x,b)$ iff $\mathrm{S}^{A,C}_{a,c}(a) \bullet d \leq R(a,b)$, i.e. $c \bullet d \leq R(a,b)$ for all $d \in D$. But it means that $\{d \in D : (\forall a \in A)\mathrm{S}^{A,C}_{a,c}(a) \bullet d \leq R(a,b)\} = \{d \in D : c \bullet d \leq R(a,b)\}$ and $\swarrow(\mathrm{S}^{A,C}_{a,c})(b) = \sup\{d \in D : (\forall x \in A)\mathrm{S}^{A,C}_{a,c}(x) \bullet d \leq R(x,b)\} = \sup\{d \in D : c \bullet d \leq R(a,b)\}$, q.e.d..

The proof of the part b) is analogous.

Firstly we prove the theorem for the case $V = \mathfrak{L}$: Define

$$\alpha_{\mathfrak{L}}(a,c) = \langle \swarrow(\mathrm{S}^{A,C}_{a,c}), \nearrow(\swarrow(\mathrm{S}^{A,C}_{a,c}))\rangle,$$

$$\beta_{\mathfrak{L}}(b,d) = \langle \swarrow(\nearrow(\mathrm{S}^{B,D}_{b,d})), \nearrow(\mathrm{S}^{B,D}_{b,d})\rangle.$$

Obviously both results are really concepts.

Claim 1

a) $\alpha_{\mathfrak{L}}$ is non-increasing in the second argument.

b) $\beta_{\mathfrak{L}}$ is non-decreasing in the second argument.

Proof. Let $c_1 \leq c_2$ are from C, we want to prove that $\alpha_{\mathfrak{L}}(a, c_1) \geq \alpha_{\mathfrak{L}}(a, c_2)$ (for an arbitrary $a \in A$). It is enough to prove that $\swarrow(\mathrm{S}^{A,C}_{a,c_1}) \geq \swarrow(\mathrm{S}^{A,C}_{a,c_2})$, i.e. $\swarrow(\mathrm{S}^{A,C}_{a,c_1})(b) \geq \swarrow(\mathrm{S}^{A,C}_{a,c_2})(b)$ for all $b \in B$, which (by the previous lemma) means $\sup\{d \in D : c_1 \bullet d \leq R(a, b)\} \geq \sup\{d \in D : c_2 \bullet d \leq R(a, b)\}$.

Because of monotoneity of $\leq$ in the first argument we have $c_1 \bullet d \leq c_2 \bullet d$, so $c_2 \bullet d \leq R(a, b)$ implies $c_1 \bullet d \leq R(a, b)$. It means that $\{d \in D : c_2 \bullet d \leq R(a, b)\} \subseteq \{d \in D : c_1 \bullet d \leq R(a, b)\}$, which implies wanted $\sup\{d \in D : c_2 \bullet d \leq R(a, b)\} \leq \sup\{d \in D : c_1 \bullet d \leq R(a, b)\}$, q.e.d..

The proof of the part b) is symmetrical.

Claim 2

$$\alpha_{\mathfrak{L}}(a, c) \geq \beta_{\mathfrak{L}}(b, d) \qquad \textit{if and only if} \qquad c \bullet d \leq R(a, b).$$

Proof. If $c \bullet d \leq R(a, b)$, then clearly $c \in \{k \in C : k \bullet d \leq R(a, b)\}$. It follows that $\mathrm{S}^{A,C}_{a,c}(a) = c \leq \sup\{k \in C : k \bullet d \leq R(a, b)\} = \nearrow(\mathrm{S}^{B,D}_{b,d})(a)$. For $x \neq a$ we have obvious $\mathrm{S}^{A,C}_{a,c}(x) = 0_C \leq \nearrow(\mathrm{S}^{B,D}_{b,d})(x)$, hence $\mathrm{S}^{A,C}_{a,c} \leq \nearrow(\mathrm{S}^{B,D}_{b,d})$. Because of the Galois connection between $\swarrow$ and $\nearrow$ it follows that $\swarrow(\mathrm{S}^{A,C}_{a,c}) \geq \swarrow(\nearrow(\mathrm{S}^{B,D}_{b,d}))$, i.e. $\alpha_{\mathfrak{L}}(a, c) \geq \beta_{\mathfrak{L}}(b, d)$.

Inversely, if $x \in A$, clearly $u \in \{k \in C : k \bullet d \leq R(x, b)\}$ iff $u \bullet d \leq R(x, b)$. Hence (using left-continuousness of $\bullet$ in the second argument) $\nearrow(\mathrm{S}^{B,D}_{b,d})(x) \bullet d = \sup\{k \in C : k \bullet d \leq R(x, b)\} \bullet d \leq R(x, b)$. So $(\forall x \in A) \nearrow(\mathrm{S}^{B,D}_{b,d})(x) \bullet d \leq R(x, b)$, and this implies $d \in \{m \in D : (\forall x \in A) \nearrow(\mathrm{S}^{B,D}_{b,d})(x) \bullet m \leq R(x, b)\}$, hence $d \leq \sup\{m \in D : (\forall x \in A) \nearrow(\mathrm{S}^{B,D}_{b,d})(x) \bullet m \leq R(x, b)\} = \swarrow(\nearrow(\mathrm{S}^{B,D}_{b,d}))(b)$ (by the definition of $\nearrow$). Because $\alpha_{\mathfrak{L}}(a, c) \geq \beta_{\mathfrak{L}}(b, d)$, i.e. $\swarrow(\mathrm{S}^{A,C}_{a,c}) \geq \swarrow(\nearrow(\mathrm{S}^{B,D}_{b,d}))$, we obtain $d \leq \swarrow(\nearrow(\mathrm{S}^{B,D}_{b,d}))(b) \leq \swarrow(\mathrm{S}^{A,C}_{a,c})(b)$. Because $w \in \{m \in D : c \bullet m \leq R(a, b)\}$ iff $c \bullet w \leq R(a, b)$, by left-continuousness of $\bullet$ in the second argument we have $c \bullet \sup\{m \in D : c \bullet m \leq R(a, b)\} \leq R(a, b)$, i.e. (by the previous lemma) $c \bullet \swarrow(\mathrm{S}^{A,C}_{a,c})(b) \leq R(a, b)$. Because of monotonity of $\bullet$ in the second argument we obtain $c \bullet d \leq c \bullet \swarrow(\mathrm{S}^{A,C}_{a,c})(b) \leq R(a, b)$, q.e.d..

Claim 3 *Let $f : A \to C$ and $g : B \to D$ and $\langle g, f \rangle$ be a concept.*

a) $\langle g, f \rangle = \inf\{\alpha_{\mathfrak{L}}(a, f(a)) : a \in A\}$.
b) $\langle g, f \rangle = \sup\{\beta_{\mathfrak{L}}(b, g(b)) : b \in B\}$.

Proof. Let $a \in A$. Then $\mathrm{S}^{A,C}_{a,f(a)}(a) = f(a)$ and $\mathrm{S}^{A,C}_{a,f(a)}(x) = 0_C \leq f(x)$ for the other $x \neq a$, hence $\mathrm{S}^{A,C}_{a,f(a)} \leq f$. Because $\swarrow$ and $\nearrow$ form a Galois connection and f is the intent of some concept, we obtain $\nearrow(\swarrow(\mathrm{S}^{A,C}_{a,f(a)})) \leq \nearrow(\swarrow(f)) = f$, so $\langle g, f \rangle \leq \langle \swarrow(\mathrm{S}^{A,C}_{a,f(a)}), \nearrow(\swarrow(\mathrm{S}^{A,C}_{a,f(a)})) \rangle = \alpha_{\mathfrak{L}}(a, f(a))$. This is true for all $a \in A$, so $\langle g, f \rangle \leq \inf\{\alpha_{\mathfrak{L}}(a, f(a)) : a \in A\}$.

Conversely, by the part 1) of the theorem we have $\inf\{\alpha_{\mathfrak{L}}(a, f(a)) : a \in A\} = \bigwedge_{a\in A} \alpha_{\mathfrak{L}}(a, f(a)) = \bigwedge_{a\in A}\langle \swarrow(\mathrm{S}^{A,C}_{a,f(a)}), \nearrow(\swarrow(\mathrm{S}^{A,C}_{a,f(a)}))\rangle = \langle \bigwedge_{a\in A} \swarrow(\mathrm{S}^{A,C}_{a,f(a)}), \nearrow(\swarrow(\bigvee_{a\in A} \nearrow(\swarrow(\mathrm{S}^{A,C}_{a,f(a)}))))\rangle$. On the other hand, using the property of composition $\swarrow$ and $\nearrow$ (following from a Galois connection between them) we have $\mathrm{S}^{A,C}_{a,f(a)} \leq \nearrow(\swarrow(\mathrm{S}^{A,C}_{a,f(a)})) = \sup\{\nearrow(\swarrow(\mathrm{S}^{A,C}_{x,f(x)})) : x \in A\}$ for all $a \in A$. It means (using lemma 1) that $f = \sup\{\mathrm{S}^{A,C}_{a,f(a)} : a \in A\} \leq \sup\{\nearrow(\swarrow(\mathrm{S}^{A,C}_{x,f(x)})) : x \in A\} \leq \nearrow(\swarrow(\sup\{\nearrow(\swarrow(\mathrm{S}^{A,C}_{x,f(x)})) : x \in A\})) = \nearrow(\swarrow(\bigvee_{x\in A} \nearrow(\swarrow(\mathrm{S}^{A,C}_{x,f(x)}))))$, i.e. $f \leq \nearrow(\swarrow(\bigvee_{a\in A} \nearrow(\swarrow(\mathrm{S}^{A,C}_{a,f(a)}))))$. Hence $\langle g, f\rangle \geq \inf\{\alpha_{\mathfrak{L}}(a, f(a)) : a \in A\}$, q.e.d..

The proof of the part b) is analogous.

Claim 4 *a)* $\alpha_{\mathfrak{L}}[A \times C]$ *is infimum-dense in* $\mathfrak{L}$.
b) $\beta_{\mathfrak{L}}[B \times D]$ *is supremum-dense in* $\mathfrak{L}$.

Proof. They are obvious consequences of the previous claim.

We have proven the second part of the basic theorem for the special case $V = \mathfrak{L}$. Now we will prove it for another cases.

First we assume that V is a complete lattice isomorphic to our $\mathfrak{L}$. Let $\varphi : \mathfrak{L} \to V$ is a witness isomorphism. Define

$$\alpha(a, c) = \varphi(\alpha_{\mathfrak{L}}(a, c)),$$

$$\beta(b, d) = \varphi(\beta_{\mathfrak{L}}(b, d)).$$

We prove that these mapping fulfill appropriate properties:

Claim 5
a) α *is non-increasing in the second argument.*
b) β *is non-decreasing in the second argument.*

Proof. If $c_1 \leq c_2$, we have (by the claim 1) $\alpha_{\mathfrak{L}}(a, c_1) \geq \alpha_{\mathfrak{L}}(c, a_2)$. Because φ is an isomorphism, we obtain $\varphi(\alpha_{\mathfrak{L}}(a, c_1)) \geq \varphi(\alpha_{\mathfrak{L}}(c, a_2))$, i.e. $\alpha(a, c_1) \geq \alpha(c, a_2)$, q.e.d..

The proof of the part b) is analogous.

Claim 6
a) $\alpha[A \times C]$ *is infimum-dense.*
b) $\beta[B \times D]$ *is supremum-dense.*

Proof. If $v \in V$, let $\langle g, f\rangle \in \mathfrak{L}$ s.t. $\varphi(\langle g, f\rangle) = v$ (such v exists, because φ is a bijection). Then $\inf\{\alpha(a, f(a)) : a \in A\} = \inf\{\varphi(\alpha_{\mathfrak{L}}(a, f(a))) : a \in A\} = \varphi(\inf\{\alpha_{\mathfrak{L}}(a, f(a)) : a \in A\})$ (because φ is an isomorphism), $= \varphi(\langle g, f\rangle)$ (by claim 2) $= v$. Hence we can express an arbitrary element of V as the infimum of some subset of $\alpha[A \times C]$, i.e. $\alpha[A \times C]$ is infimum-dense, q.e.d..

The fact that $\beta[B \times D]$ is supremum-dense can be proven analogously.

Claim 7 *$\alpha(a,c) \geq \beta(b,d)$ if and only if $c \bullet d \leq R(a,b)$.*

Proof. $\alpha(a,c) \geq \beta(b,d)$ i.e. $\varphi(\alpha_{\mathfrak{L}}(a,c)) \geq \varphi(\beta_{\mathfrak{L}}(b,d))$ iff $\alpha_{\mathfrak{L}}(a,c) \geq \beta_{\mathfrak{L}}(b,d)$ (because φ is an isomorphism), iff $c \bullet d \leq R(a,b)$ (we have proven it above), q.e.d..

Now we have proven the first direction, we are going to show the opposite one. Let α and β have appropriate properties.

Claim 8 *a) For all $a \in A$, $c \in C$ and $v \in V$*

$$\alpha(a,c) \geq v \qquad \textit{iff} \qquad (\forall b \in B)(\forall d \in D)(\beta(b,d) \leq v \rightarrow \alpha(a,c) \geq \beta(b,d)).$$

b) For all $b \in B$, $d \in D$ and $v \in V$

$$\beta(b,d) \leq v \qquad \textit{iff} \qquad (\forall a \in A)(\forall c \in C)(\alpha(a,c) \geq v \rightarrow \alpha(a,c) \geq \beta(b,d)).$$

Proof. If $\alpha(a,c) \geq v$, then clearly $\beta(b,d) \leq v$ implies $\alpha(a,c) \geq \beta(b,d)$ what means $c \bullet d \leq R(a,b)$ by assumption on α and β.

Conversely, using supremum-density of $\beta[B \times D]$ our v can be expressed in a form $v = \sup\{\beta(b_i,d_i) : i \in I\}$ for some set of pairs $\{\langle b_i,d_i\rangle : i \in I\}$. This implies that $\beta(b_i,d_i) \leq v$ for all $i \in I$, hence (by our assumption) $\alpha(a,c) \geq \beta(b_i,d_i)$ for all $i \in I$. But it means that $\alpha(a,c) \geq \sup\{\beta(b_i,d_i) : i \in I\} = v$, q.e.d..

The proof of the part b) is symmetrical.

Now we can define the following function $\varphi : \mathfrak{L} \rightarrow V$ in the following way:

$$\varphi(\langle g,f\rangle) = \inf\{\alpha(a,f(a)) : a \in A\}$$

and show that it is a wanted isomorphism.

Claim 9 *φ is order-preserving.*

Proof. $\langle g_1,f_1\rangle \leq \langle g_2,f_2\rangle$ iff $f_1 \geq f_2$ iff $f_1(a) \geq f_2(a)$ for all $a \in A$. Because α is non-increasing in the second argument, it implies $\alpha(a,f_1(a)) \leq \alpha(a,f_2(a))$. So we have $\varphi(\langle g_1,f_1\rangle) = \inf\{\alpha(x,f_1(x)) : x \in A\} \leq \alpha(a,f_2(a))$ for all $a \in A$ and (using the definition of an infimum) $\varphi(\langle g_1,f_1\rangle) \leq \inf\{\alpha(a,f_2(a)) : a \in A\} = \varphi(\langle g_2,f_2\rangle)$, q.e.d..

Now we define the function $\psi : \mathfrak{L} \rightarrow V$ by

$$\psi(v) = \langle g_v,f_v\rangle \quad \text{such that}$$

$$f_v(a) = \sup\{c \in C : \alpha(a,c) \geq v\},$$

$$g_v(b) = \sup\{d \in D : \beta(b,d) \leq v\}$$

and show that $\psi = \varphi^{-1}$.

First we prove that $\langle g_v,f_v\rangle$ is really concept:

Claim 10 *a)* $\nearrow(g_v) = f_v$.
b) $\swarrow(f_v) = g_v$.

Proof. $\alpha(a,c) \geq v$ is (by previous lemma) equivalent to $(\forall b \in B)(\forall d \in D)(\beta(b,d) \leq v \to \alpha(a,c) \geq \beta(b,d))$ and because of the relation between α, β and R, this is equivalent to $(\forall b \in B)(\forall d \in D)(\beta(b,d) \leq v \to c \bullet d \leq R(a,b))$. This can be replaced equivalently by $(\forall b \in B) c \bullet \sup\{d \in D : \beta(b,d) \leq v\} \leq R(a,b)$ (one implication follows from the definition of a supremum, the second one from left-continuity of $\bullet$), i.e. $(\forall b \in B) c \bullet g_v(b) \leq R(a,b)$. This gives the equality $\sup\{c \in C : \alpha(a,c) \geq v\} = \sup\{c \in C : (\forall b \in B) c \bullet g_v(b) \leq R(a,b)\}$, i.e. $f_v(a) = \nearrow(g_v)(a)$ (by the definitions of f_v and $\nearrow$) for all $a \in A$. It means that $f_v = \nearrow(g_v)$, q.e.d..

The proof of the part b) is symmetrical.

Claim 11 ψ *is order-preserving.*

Proof. If $v_1 \leq v_2$ then clearly $\alpha(a,c) \geq v_2$ implies $\alpha(a,c) \geq v_1$. It follows that $\{c \in C : \alpha(a,c) \geq v_2\} \subseteq \{c \in C : \alpha(a,c) \geq v_1\}$, i.e. $f_{v_2}(a) = \sup\{c \in C : \alpha(a,c) \geq v_2\} \leq \sup\{c \in C : \alpha(a,c) \geq v_1\} = f_{v_1}(a)$ for all $a \in A$. This means that $f_{v_1} \geq f_{v_2}$, i.e. $\psi(v_1) = \langle g_{v_1}, f_{v_1} \rangle \leq \langle g_{v_2}, f_{v_2} \rangle = \psi(v_2)$, q.e.d..

Claim 12 $\varphi(\psi(v)) = v$.

Proof. Using supremum-density of $\beta[B \times D]$ our v can be expressed in a form $v = \sup\{\beta(b_i,d_i) : i \in I\}$ for some set of pairs $\{\langle b_i, d_i \rangle : i \in I\}$. This implies that $\beta(b_i,d_i) \leq v$ for all $i \in I$. If $a \in A$ and $c \in C$ are such that $\alpha(a,c) \geq v$, we have $\beta(b_i,d_i) \leq \alpha(a,c)$ for all $i \in I$. Hence $c \bullet d_i \leq R(a,b_i)$, and using the left-continuousness of $\bullet$ in the first argument we obtain $\sup\{c \in C : \alpha(a,c) \leq v\} \bullet d_i \leq R(a,b_i)$, i.e. (by the definition of f_v) $f_v(a) \bullet d_i \leq R(a,b_i)$ for all $i \in I$ and $a \in A$. But it means that $\alpha(a, f_v(a)) \geq \beta(b_i,d_i)$ and this implies $\alpha(a, f_v(a)) \geq \sup\{\beta(b_i,d_i) : i \in I\} = v$. Hence $\psi(\langle g_v, f_v \rangle) = \inf\{\alpha(a, f_v(a)) : a \in A\} \geq v$.

Conversely, using supremum-density of $\alpha[A \times C]$ our v can be expressed in a form $v = \inf\{\alpha(a_i,c_i) : i \in I\}$ for some set of pairs $\{\langle a_i, c_i \rangle : i \in I\}$. Clearly $\alpha(a_i,c_i) \geq v$, therefore $c_i \in \{c \in C : \alpha(a_i,c) \geq v\}$, which implies $c_i \leq \sup\{c \in C : \alpha(a_i,c) \geq v\} = f_v(a_i)$, for all $i \in I$. Because α is non-increasing in the second argument, we obtain $\alpha(a_i,c_i) \geq \alpha(a_i, f_v(a_i))$ for all $i \in I$. And because $\alpha(a_i, f_v(a_i)) \in \{\alpha(a, f_v(a)) : a \in A\}$, we have $\alpha(a_i,c_i) \geq \alpha(a_i, f_v(a_i)) \geq \inf\{\alpha(a, f_v(a)) : a \in A\} = \psi(\langle g_v, f_v \rangle)$, for all $i \in I$. This implies $v = \inf\{\alpha(a_i,c_i) : i \in I\} \geq \psi(\langle g_v, f_v \rangle)$, q.e.d..

Claim 13 $\psi(\varphi(\langle g, f \rangle)) = \langle g, f \rangle$.

Proof. Let $v = \varphi(\langle g, f \rangle) = \inf\{\alpha(a, f(a)) : a \in A\}$, it is enough to prove that $g = g_v$.

Because $(\forall a \in A)(f(a) \bullet d \leq R(a,b))$ iff $(\forall a \in A)(\beta(b,d) \leq \alpha(a, f(a)))$ (it is a property of α and β), and this is (by the definition of a infimum) equivalent to $\beta(b,d) \leq \inf\{\alpha(a, f(a)) : a \in A\} = v$, we obtain $\{d \in D : (\forall a \in A) f(a) \bullet d \leq$

$R(a,b)\} = \{d \in D : \beta(b,d) \leq v\}$, which implies (using the definitions of $\swarrow$ and g_v) $\swarrow(f)(b) = \sup\{d \in D : (\forall a \in A) f(a) \bullet d \leq R(a,b)\} = \sup\{d \in D : \beta(b,d) \leq v\} = g_v(b)$. It is true for all $b \in B$, so $g = \swarrow(f) = g_v$, q.e.d..

To conclude, we prove that both φ and ψ are order-preserving and both $\varphi \circ \psi$ and $\psi \circ \varphi$ are identities. All this means that really $\psi = \varphi^{-1}$ and both $\varphi : V \to \mathfrak{L}$ and $psi : \mathfrak{L} \to V$ are isomorphisms. Quod erat demonstrandum.

As a bonus we prove next property of our mappings:

Claim 14 *a)*

$$\alpha(a, \sup\{c_i : i \in I\}) = \inf\{\alpha(a, c_i) : i \in I\}.$$

b)

$$\beta(b, \sup\{d_i : i \in I\}) = \sup\{\alpha(b, d_i) : i \in I\}.$$

Proof. $\beta[B \times D]$ is supremum-dense, so $\alpha(a, \sup\{c_i : i \in I\} = \sup\{\beta(b_j, d_j) : j \in J\}$ for some $\{\langle b_j, d_j\rangle : j \in J\}$. This implies $\alpha(a, \sup\{c_i : i \in I\} \geq \beta(b_j, d_j)$, i.e. (by the definition) $\sup\{c_i : i \in I\} \bullet d_j \leq R(a, b_j)$ for all $j \in J$. It follows that $c_i \bullet d_j \leq R(a, b_j)$, i.e. $\alpha(a, c_i) \geq \beta(b_j, d_j)$ for all $i \in I$, $j \in J$. By the definitions of a infimum and a supremum we obtain $\inf\{\alpha(a, c_i) : i \in I\} \geq \beta(b_j, d_j)$ and $\inf\{\alpha(a, c_i) : i \in I\} \geq \sup\{\beta(b_j, d_j) : j \in J\} = \alpha(a, \sup\{c_i : i \in I\}$.

Conversely again from the supremum-density of $\beta[B \times D]$, we have $\inf\{\alpha(a, c_i) : i \in I\} = \sup\{\beta(b_j, d_j) : j \in J\}$ for some $\{\langle b_j, d_j\rangle : j \in J\}$. It follows that $\alpha(a, c_i) \geq \sup\{\beta(b_j, d_j) : j \in J\}$ for all $j \in J$ and $\alpha(a, c_i) \geq \beta(b_j, d_j)$, i.e. $c_i \bullet d_j \leq R(a, b_j)$ for all $i \in I$, $j \in J$. Again, it is equivalent $\sup\{c_i : i \in I\} \bullet d_j \leq R(a, b_j)$, i.e. $\alpha(a, \sup\{c_i : i \in I\}) \geq \beta(b_j, d_j)$ for all $j \in J$, hence $\alpha(a, \sup\{c_i : i \in I\}) \geq \sup\{\beta(b_j, d_j) : j \in J\} = \inf\{\alpha(a, c_i) : i \in I\}$, q.e.d..

The proof of the part b) is symmetrical, we use the infinum-density of the set $\alpha[A \times C]$.

Yet one note on an asymmetry in the definition of φ. If we define the function $\varphi' : \mathfrak{L} \to V$ in the following way (symmetrically to φ):

$$\varphi'(\langle g, f\rangle) = \sup\{\beta(b, g(b)) : b \in B\},$$

it can be proven (analogously to the claim 12) that $\varphi'(\psi(v)) = v$. So for arbitrary concept $\langle g, f\rangle$ we take $v = \psi^{-1}(\langle g, f\rangle)$ and we obtain $\varphi'(\langle g, f\rangle) = \varphi'(\psi(v)) = v = \varphi(\psi(v)) = \varphi(\langle g, f\rangle)$, i.e. $\varphi' = \varphi$. Hence our asymmetry in the definition of φ is only seeming and

$$\inf\{\alpha(a, f(a)) : a \in A\} = \sup\{\beta(b, g(b)) : b \in B\}.$$

3 Conclusions

In this paper we prove the extended version of the basic theorem on a generalized concept lattice, which is a common platform for till now known fuzzifications of a classical crisp concept lattice. The main idea of it is to use two different complete lattices, one for objects and maybe another for attributes.

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